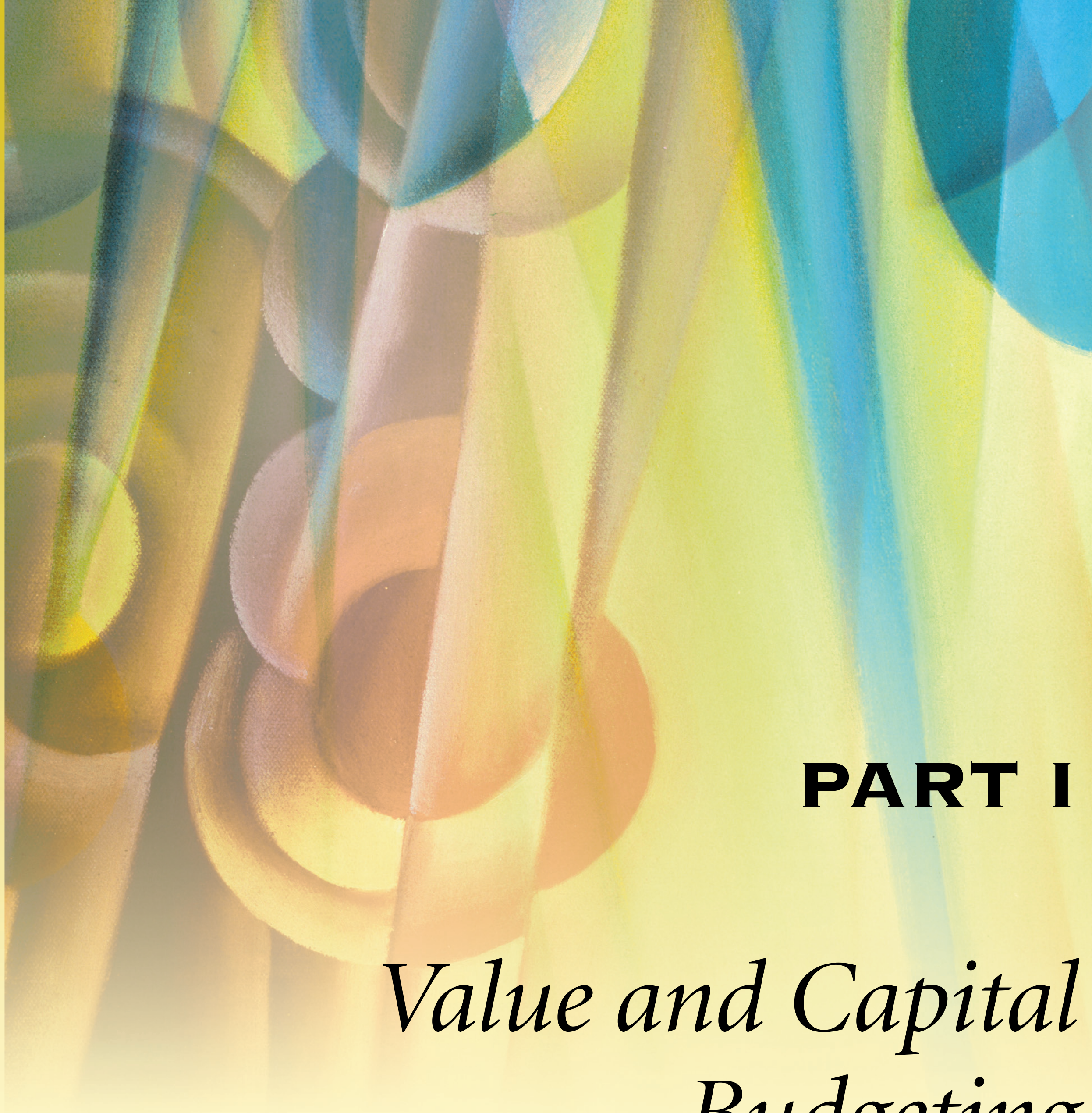

PART I

Value and Capital Budgeting

IN A PERFECT MARKET UNDER RISK NEUTRALITY

The two primary goals of this first part of the book (Chapters 2–6) are to explain how to work with rates of return and how to decide whether to accept or reject investment projects. We assume in this part that there are no taxes, no transaction costs, no disagreements, and no limits as to the number of sellers and buyers in the market. This is the so-called perfect market. I will explain later why a perfect market makes your life a lot easier.

WHAT YOU WANT TO LEARN IN THIS PART

- In Chapter 2, we start with the simplest possible scenario. In addition to the perfect market, we assume that there is no uncertainty: You know everything. And we assume that all rates of return in the economy are the same: A 1-year investment pays the same and perfectly known rate of return per annum as a 10-year investment. Under these assumptions, you learn how 1-year returns translate into multiyear returns and when you should accept or reject a project. The chapter introduces the important concept of "present value."

  Typical questions: If you earn 5% per year, how much will you earn over 10 years? If you earn 100% over 10 years, how much will you earn per year? What is the value of a project that will deliver $1,000,000 in 10 years? Should you buy this project if it cost you $650,000? What inputs do you need to decide this?

- In Chapter 3, you learn how to value particular kinds of projects—perpetuities and annuities—if the economy-wide interest rate remains constant. You then learn how to apply the formulas to the valuation of stocks and bonds. The popular Gordon dividend growth model for valuing stocks assumes that dividends are a simple growing perpetuity cash flow stream, which makes it a perfect application of the perpetuity formula. Mortgages and other bonds are good applications of pricing using the annuities formulas.

  Typical questions: If a firm pays $1/share dividends next year, growing by 3% per year forever, then what should its stock price be? What is the monthly payment for a $300,000 mortgage bond if the interest rate is 4% per year?

- In Chapter 4, you learn more about capital budgeting methods. Although net present value (NPV) is the correct method, at least one other common method often comes to the correct result: the internal rate of return. In the real world, a number of other, plainly incorrect, methods are also in wide use. You should know why you should be wary of them. This chapter also tells you what CFOs actually rely on.

  Typical questions: If a project has one investment outflow and two return inflows, how would you compute a "rate of return"? Can you accept projects whose rates of return are above their cost of capital? How bad is it when you use incorrect estimates—as you inevitably will—in your calculations? What are the big problems with a rule that accepts those projects that return money most quickly?

- In Chapter 5, you abandon the assumption that annual rates of return are the same regardless of the length of time of your investment. For example, 1-year investments may pay 2% per year, while 10-year investments may pay 5% per year. The scenario of time-varying rates of return is more realistic, but the questions that you want to answer still remain the same as those in Chapter 2. (The chapter then also explains more advanced aspects of bonds, such as the Treasury yield curve.)

  Typical questions: If you earn 5% in the first year and 10% in the second year, how much will you earn over both years? What is the meaning of a 4% annualized interest rate? What is the meaning of a 4% yield-to-maturity? How can you value projects if appropriate rates of return depend on different time horizons?

- In Chapter 6, you abandon the assumption that you know the future. To be able to study uncertainty in the real world, you must first learn how to describe it. This is done with statistics, the necessary aspects of which are explained here, too. The chapter then introduces risk neutrality, which is an assumption that can make it easier to understand some concepts in finance under uncertainty. Perhaps the two most important concepts are the difference between promised and expected rates of return and the difference between debt and equity. Under uncertainty, a project may not return the promised amount. Because of the possibility of default, the *stated* rate of return must be higher than the *expected* rate of return. Although you are interested in the latter, it is almost always only the former that you are quoted (promised). It is important that you always draw a sharp distinction between promised (stated) rates of return and expected rates of return. The second concept that this chapter explains is the difference between debt and equity—corporate claims that have a meaningful difference only under uncertainty.

Typical questions: If there is a 2% chance that your borrower will not return the money, how much extra interest should you charge? From an investment perspective, what is the difference between debt and equity? What is financing priority? What is a residual claim?

Looking ahead, Part II will continue with uncertainty scenarios in which investors are risk averse. Part III will explain what happens when financial markets or decision rules are not perfect.

The Time Value of Money and Net Present Value

THE MOTHER OF ALL FINANCE

We begin with the concept of a rate of return—the cornerstone of finance. You can always earn interest by depositing your money today into the bank. This means that money today is more valuable than the same amount of money next year. This concept is called the *time value of money*—$1 in present value is better than $1 in future value.

Investors make up just one side of the financial markets. They give money today in order to receive money in the future. Firms make up the other side. The process firms use to decide what to do with their money—which projects to take and which projects to pass up—is called *capital budgeting*. You will learn that there is one clear best method for making this critical decision. The firm should translate all *future* cash flows—both inflows and outflows—into their equivalent *present values* today, and then add them up to find the *net present value*, or NPV. The firm should take all projects that have positive net present values and reject all projects that have negative net present values.

This all sounds more complex than it is, so we'd better get started.

2.1 OUR BASIC SCENARIO: PERFECT MARKETS, CERTAINTY, CONSTANT INTEREST RATES

We start with a so-called perfect market.

As promised, we begin with the simplest possible scenario. In finance, this means that we assume that we are living in a so-called **perfect market**:

- There are no taxes.
- There are no transaction costs (costs incurred when buying and selling).

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- There are no differences in information or opinions among investors.
- There are so many buyers and sellers (investors and firms) in the market that the presence or absence of just one (or a few) individuals does not have an influence on the price.

The perfect market allows us to focus on the basic concepts in their purest forms, without messy real-world factors complicating the exposition. We will use these assumptions as our sketch of how financial markets operate, though not necessarily how firms' product markets work. You will learn in Chapter 10 how to operate in a world that is not perfect. (This will be a lot messier.)

In early chapters only, we add even stronger assumptions.

In this chapter, we will make three additional assumptions (that are not required for a market to be considered "perfect") to further simplify the world:

- There is no risk or uncertainty. You have perfect foresight.
- There is no inflation.
- The interest rate per period is the same.

Of course, this financial utopia is unrealistic. However, the tools that you will learn in this chapter will also work in later chapters, where the world becomes not only progressively more realistic but also more difficult. Conversely, if any tool does not give the right answer in our simple world, it would surely make no sense in a more realistic world.

SOLVE NOW!

Q 2.1 What are the four perfect market assumptions?

2.2 LOANS AND BONDS

Finance jargon: interest, loan, bond, fixed income, maturity.

The material in this chapter is easiest to explain in the context of bonds and loans. A **loan** is the commitment of a borrower to pay a predetermined amount of cash at one or more predetermined times in the future (the final one called **maturity**), usually in exchange for cash up front today. Loosely speaking, the difference between the money lent and the money paid back is the **interest** that the lender earns. A **bond** is a particular kind of loan, so named because it "binds" the borrower to pay money. Thus, for an investor, "buying a bond" is the same as "extending a loan." Bond buying is the process of giving cash today and receiving a promise for money in the future. Similarly, from the firm's point of view, it is "giving a bond," "issuing a bond," or "selling a bond." Loans and bonds are also sometimes called **fixed income**, because they "promise" a fixed amount of payments to the holder of the bond.

Why learn bonds first? Because they are easiest.

You should view a bond as just another type of investment project—money goes in, and money comes out. In Chapter 5, you will learn more about Treasuries, which are bonds issued by the U.S. Treasury. The beauty of such bonds is that you know what the cash flows will be. Besides, much more capital in the economy is tied up in bonds and loans than is tied up in stocks, so understanding bonds well is very useful in itself.

Interest rates: limited upside. Rates of return: arbitrary upside.

You already know that the net return on a loan is called interest, and that the rate of return on a loan is called the **interest rate**—though we will soon firm up your knowledge about interest rates. One difference between an interest payment and a noninterest payment is that the former usually has a maximum payment, whereas the latter can have unlimited upside potential. However, not every rate of return is an interest rate. For example, an investment in a lottery ticket is not a loan, so it does not offer an interest rate, just a rate of return. In real life, its payoff is uncertain—it could be anything from zero to an unlimited amount. The same applies to stocks and many corporate projects. Many of our examples use the phrase "interest rate," even though the examples almost always work for any other rates of return, too.

Bond: defined by payment next year. Savings: defined by deposit this year.

Is there any difference between buying a bond for $1,000 and putting $1,000 into a bank savings account? Yes, a small one. The bond is defined by its future promised payoffs—say, $1,100 next year—and the bond's value and price today are based on these future payoffs. But as the bond owner, you know exactly how much you will receive next year. An investment in a bank savings account is defined by its investment today. The interest rate can and will change every day, so you do not know what you will end up with next year. The exact amount depends on future interest rates. For example, it could be $1,080 (if interest rates decrease) or $1,120 (if interest rates increase).

A bank savings account is like a sequence of 1-day bonds.

If you want, you can think of a savings account as a sequence of consecutive 1-day bonds: When you deposit money, you buy a 1-day bond, for which you know the interest rate this one day in advance, and the money automatically gets reinvested tomorrow into another bond with whatever the interest rate will be tomorrow.

SOLVE NOW!

Q 2.2 Is a deposit into a savings account more like a long-term bond investment or more like a series of short-term bond investments?

2.3 RETURNS, NET RETURNS, AND RATES OF RETURN

Defining return and our time. Our convention is that 0 means "right now."

The most fundamental financial concept is that of a return. The payoff or (dollar) **return** of an investment is simply the amount of cash (C) it returns. For example, an investment project that returns $12 at time 1 has

$$\text{Return at Time 1} = \$12$$
$$\text{Return}_1 = C_1$$

The subscript is an instant in time, usually abbreviated by the letter t. When exactly time 1 occurs is not important: It could be tomorrow, next month, or next year. But if we mean "right now," we use the subscript 0.

Defining net return and rate of return.

The net payoff, or **net return**, is the difference between the return and the initial investment. It is positive if the project is profitable and negative if it is unprofitable. For example, if the investment costs $10 today and returns $12 at time 1 with nothing in between, then it earns a net return of $2. Notation-wise, we really should use two subscripts on returns—the time when the investment starts (0) and when it ends (1). This would make it something like "Net Return$_{0,1}$." Yikes! Let's just omit the first subscript on such flows when it is zero.

$$\text{Net Return from Time 0 to Time 1} = \$12 - \$10 = \$2$$
$$\text{Net Return}_1 = C_1 - C_0$$

The **rate of return** is the net return expressed as a percentage of the initial investment.

$$\begin{array}{c}\text{Rate of Return}\\ \text{from Time 0 to Time 1}\end{array} = \frac{\$2}{\$10} = 20\%$$
$$r_1 = \frac{\text{Net Return from Time 0 to Time 1}}{\text{Purchase Price at Time 0}}$$

Often, it is convenient to calculate this as

$$r_1 = \frac{\$12 - \$10}{\$10} = \frac{\$12}{\$10} - 1 = 20\%$$
$$r_1 = \frac{C_1 - C_0}{C_0} = \frac{C_1}{C_0} - 1 \qquad (2.1)$$

Rates of return are used so often that they have their own unique letter, r. Percent (the symbol %) is a unit of 1/100. 20% is the same as 0.20.

How to compute returns with interim payments. Capital gains versus returns.

Many investments have interim payments. For example, many stocks pay interim cash **dividends**, many bonds pay interim cash **coupons**, and many real estate investments pay interim rent. How would you calculate the rate of return then? One simple method is to just add interim payments to the numerator. Say an investment costs \$92, pays a dividend of \$5 (at the end of the period), and then is worth \$110. Its rate of return is

$$r = \frac{\$110 + \$5 - \$92}{\$92} = \frac{\$110 - \$92}{\$92} + \frac{\$5}{\$92} = 25\%$$
$$r_1 = \frac{C_1 + \text{All Dividends from 0 to 1} - C_0}{C_0} = \underbrace{\frac{C_1 - C_0}{C_0}}_{\text{Percent Price Change}} + \underbrace{\frac{\text{All Dividends}}{C_0}}_{\text{Dividend Yield}}$$

When there are intermittent payments and final payments, then returns are often broken down into two additive parts. The first part, the price change or **capital gain**, is the difference between the purchase price and the final price, *not* counting interim payments. Here, the capital gain is the difference between \$110 and \$92, that is, the \$18 change in the price of the investment. It is often quoted in percent of the price, which would be \$18/\$92 or 19.6% here. The second part is the amount received in interim payments. It is the dividend or coupon or rent, here \$5. When it is divided by the price, it has names like **dividend yield**, **current yield**, **rental yield**, or **coupon yield**, and these are also usually stated in percentage terms. In our example, the dividend yield is \$5/\$92 ≈ 5.4%. Of course, if the interim yield is high, you might be experiencing a negative capital gain and still have a positive rate of return. For example, a bond that costs $500, pays a coupon of $50, and then sells for $490, has a **capital loss** of $10 (which comes to a −2% capital yield) but a rate of return of ($490 + $50 − $500)/$500 = +8%. You will almost always work with rates of return, not with capital gains. The only exception is when you have to work with taxes, because the IRS treats capital gains differently from interim payments. (We will cover taxes in Section 10.4.)

➤ Taxes on capital gains, Section 10.4, p. 323

People often use incorrect terms, but the meaning is usually clear, so this is harmless.

Most of the time, people (incorrectly but harmlessly) abbreviate a rate of return or net return by calling it just a return. For example, if you say that the return on your $10,000 stock purchase was 10%, you obviously do not mean you received a unitless 0.1. You really mean that your rate of return was 10% and you received $1,000. This is usually benign, because your listener will know what you mean. Potentially more harmful is the use of the phrase *yield*, which, strictly speaking, means *rate of return*. However, it is often misused as a shortcut for dividend yield or coupon yield (the percent payout that a stock or a bond provides). If you say that the yield on your stock was 5%, then some listeners may interpret it to mean that you earned a total rate of return of 5%, whereas others may interpret it to mean that your stock paid a dividend yield of 5%.

(Nominal) interest rates are usually nonnegative.

Interest rates should logically always be positive. After all, you can always earn 0% if you keep your money under your mattress—you thereby end up with as much money next period as you have this period. Why give your money to someone today who will give you less than 0% (less money in the future)? Consequently, interest rates are indeed almost always positive—the rare exceptions being both bizarre and usually trivial.

Basis points avoid an ambiguity in the English language: 100 basis points equals 1%.

Here is another language problem: What does the statement "the interest rate has just increased by 5%" mean? It could mean either that the previous interest rate, say, 10%, has just increased from 10% to 10% · (1 + 5%) = 10.5%, or that it has increased from 10% to 15%. Because this is unclear, the **basis point** unit was invented. A basis point is simply 1/100 of a percent. If you state that your interest rate has increased by 50 basis points, you definitely mean that the interest rate has increased from 10% to 10.5%. If you state that your interest rate has increased by 500 basis points, you definitely mean that the interest rate has increased from 10% to 15%.

IMPORTANT: 100 basis points constitute 1%. Basis points avoid "percentage ambiguities."

SOLVE NOW!

Q 2.3 A project offers a return of $1,050 for an investment of $1,000. What is the rate of return?

Q 2.4 A project offers a net return of $25 for an investment of $1,000. What is the rate of return?

Q 2.5 Is 10 the same as 1,000%?

Q 2.6 You purchase a stock for $40 per share today. It will pay a dividend of $1 next month. If you can sell it for $45 right after the dividend is paid,

ANECDOTE Interest Rates over the Millennia

Historical interest rates are fascinating, perhaps because they look so similar to today's interest rates. Nowadays, typical interest rates range from 2% to 20% (depending on other factors). For over 2,500 years, from about the thirtieth century B.C.E. to the sixth century B.C.E., normal interest rates in Sumer and Babylonia hovered around 10–25% per annum, though 20% was the legal maximum. In ancient Greece, interest rates in the sixth century B.C.E. were about 16–18%, dropping steadily to about 8% by the turn of the millennium. Interest rates in ancient Egypt tended to be about 10–12%. In ancient Rome, interest rates started at about 8% in the fifth century B.C.E. but began to increase to about 12% by the third century A.C.E. (a time of great upheaval). When lending resumed in the late Middle Ages (twelfth century), personal loans in England fetched about 50% per annum, though they tended to hover between 10–20% in the rest of Europe. By the Renaissance, commercial loan rates had fallen to 5–15% in Italy, the Netherlands, and France. By the seventeenth century, even English interest rates had dropped to 6–10% in the first half, and to 3–6% in the second half. Mortgage rates tended to be lower yet. Most of the American Revolution was financed with French and Dutch loans at interest rates of 4–5%.

what would be its dividend yield, what would be its capital gain (also quoted as a capital gain yield), and what would be its total holding rate of return?

Q 2.7 If the interest rate of 9% increases to 12%, how many basis points did it increase?

Q 2.8 If the interest rate of 10% decreased by 20 basis points, what is the new interest rate?

2.4 THE TIME VALUE OF MONEY, FUTURE VALUE, AND COMPOUNDING

Because you can earn interest, a given amount of money today is worth more than the same amount of money in the future. After all, you could always deposit your money today into the bank and thereby get back more money in the future. This is an example of the concept of the **time value of money**, which says that a dollar today is worth more than a dollar tomorrow. This is one of the most basic and important concepts in finance.

2.4A THE FUTURE VALUE OF MONEY

Here is how to calculate future payoffs given a rate of return and an initial investment.

How much money will you receive in the future if the rate of return is 20% and you invest $100 today? Turn around the rate of return formula (Formula 2.1) to determine how money will grow over time given a rate of return:

$$20\% = \frac{\$120 - \$100}{\$100} \Leftrightarrow \$100 \cdot (1 + 20\%) = \$100 \cdot 1.2 = \$120$$

$$r_1 = \frac{C_1 - C_0}{C_0} \Leftrightarrow C_0 \cdot (1 + r_1) = C_1$$

The $120 next year is called the **future value** (**FV**) of $100 today. Thus, future value is the value of a present cash amount at some point in the future. It is the time value of money that causes the future value, \$120, to be higher than its present value (PV), \$100. Using the abbreviations FV and PV, you could also have written the above formula as

$$r_1 = \frac{\text{FV} - \text{PV}}{\text{PV}} \Leftrightarrow \text{FV} = \text{PV} \cdot (1 + r)$$

(If we omit the subscript on the r, it means a 1-period interest rate from now to time 1, i.e., r_1.) Please note that the time value of money is not the fact that the prices of goods may change between today and tomorrow (that would be inflation). Instead, the time value of money is based exclusively on the fact that your money can earn interest. Any amount of cash today is worth more than the same amount of cash tomorrow. Tomorrow, it will be the same amount plus interest.

➤ Section 5.2, "Inflation," p. 97

SOLVE NOW!

Q 2.9 A project has a rate of return of 30%. What is the payoff if the initial investment is \$250?

2.4B COMPOUNDING AND FUTURE VALUE

Interest on interest (or rate of return on rate of return) means rates cannot be added.

Now, what if you can earn the same 20% year after year and reinvest all your money? What would your 2-year rate of return be? Definitely *not* 20% + 20% = 40%! You know that you will have \$120 in year 1, which you can reinvest at a 20% rate of return from year 1 to year 2. Thus, you will end up with

$$\$100 \cdot (1 + 20\%)^2 = \$100 \cdot 1.2^2 = \$120 \cdot (1 + 20\%) = \$120 \cdot 1.2 = \$144$$
$$C_0 \cdot (1 + r)^2 \qquad = \qquad C_1 \cdot (1 + r) \qquad = \qquad C_2$$

This \$144—which is, of course, again a future value of \$100 today—represents a total 2-year rate of return of

$$r_2 = \frac{\$144 - \$100}{\$100} = \frac{\$144}{\$100} - 1 = 44\%$$
$$\frac{C_2 - C_0}{C_0} = \frac{C_2}{C_0} - 1 = r_2$$

This is more than 40% because the original net return of \$20 in the first year earned an additional \$4 in interest in the second year. You earn interest on interest! This is also called **compound interest**. Similarly, what would be your 3-year rate of return? You would invest \$144 at 20%, which would provide you with

$$C_3 = \$144 \cdot (1 + 20\%) = \$144 \cdot 1.2 = \$100 \cdot (1 + 20\%)^3 = \$100 \cdot 1.2^3 = \$172.80$$
$$C_3 = \quad C_2 \cdot (1 + r) \qquad = \qquad C_0 \cdot (1 + r)^3 \qquad = \qquad C_3$$

Your 3-year rate of return from time 0 to time 3, call it r_3, would thus be

$$r_3 = \frac{\$172.80 - \$100}{\$100} = \frac{\$172.80}{\$100} - 1 = 72.8\%$$

$$\frac{C_3 - C_0}{C_0} = \frac{C_3}{C_0} - 1 = r_3$$

This formula translates the three sequential 1-year rates of return into one 3-year **holding rate of return**—that is, what you earn if you hold the investment for the entire period. This process is called **compounding**, and the formula that does it is the "one-plus formula":

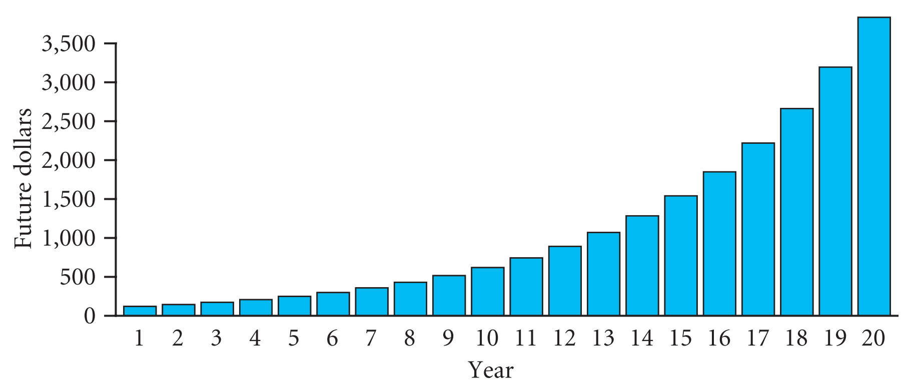
$$(1 + 72.8\%) = (1 + 20\%) \cdot (1 + 20\%) \cdot (1 + 20\%)$$

$$(1 + r_3) = (1 + r) \cdot (1 + r) \cdot (1 + r)$$

or, if you prefer it shorter,

$$1.728 = 1.2^3$$

Figure 2.1 shows how your \$100 would grow if you continued investing it at a rate of return of 20% per annum. The function is exponential—that is, it grows faster and faster as interest earns more interest.



Period	Start value	1 + one-year rate	End value	Total factor on \$100	Total rate of return $r_t = (1 + r)^t - 1$
0 to 1	\$100	(1 + 20%)	\$120.00	1.2	20.0%
1 to 2	\$120	(1 + 20%)	\$144.00	$1.2 \cdot 1.2 = 1.44$	44.0%
2 to 3	\$144	(1 + 20%)	\$172.80	$1.2 \cdot 1.2 \cdot 1.2 = 1.728$	72.8%
⋮					

Money grows at a constant rate of 20% per annum. If you compute the graphed value at 20 years out, you will find that each dollar invested right now is worth \$38.34 in 20 years. The money at first grows in a roughly linear pattern, but as more and more interest accumulates and itself earns more interest, the graph accelerates steeply upward.

FIGURE 2.1 Compounding over 20 Years at 20% per Annum

IMPORTANT: The compounding formula translates sequential future rates of return into an overall holding rate of return:

$$\underbrace{(1+r_t)}_{\text{Multiperiod Holding Rate of Return}} = \underbrace{(1+r^t)}_{\text{Multiperiod Holding Rate of Return}} = \underbrace{(1+r)}_{\text{Current 1-Period Spot Rate of Return}} \cdot \underbrace{(1+r)}_{\text{Next 1-Period Rate of Return}} \cdots \underbrace{(1+r)}_{\text{Final 1-Period Rate of Return}}$$

The first rate is called the spot rate because it starts now (on the spot).

The compounding formula is so common that you must memorize it.

Another example of a payoff computation.

You can use the compounding formula to compute all sorts of future payoffs. For example, an investment project that costs \$212 today and earns 10% each year for 12 years will yield an overall holding rate of return of

$$r_{12} = (1+10\%)^{12} - 1 = (1.1^{12} - 1) \approx 213.8\%$$
$$(1+r^t) - 1 \quad = \quad r_{12}$$

Your \$212 investment today would therefore turn into a future value of

$$C_{12} = \$212 \cdot (1+10\%)^{12} = \$212 \cdot 1.1^{12} \approx \$212 \cdot (1+213.8\%) \approx \$665.35$$
$$C_0 \cdot (1+r)^{12} \quad = \quad C_{12}$$

"Uncompounding": Turn around the formula to compute individual holding rates.

Now suppose you wanted to know what constant two 1-year interest rates (r) would give you a 2-year rate of return of 50%. It is not 25%, because $(1+25\%) \cdot (1+25\%) - 1 = 1.25^2 - 1 = 56.25\%$. Instead, you need to solve

$$(1+r) \cdot (1+r) = (1+r)^2 = 1 + 50\% = 1.50$$

The correct answer is

$$r = \sqrt[2]{1.50} - 1 \approx 22.47\%$$
$$= \sqrt[t]{1+r_t} - 1 = \quad r$$

Check your answer: $(1+22.47\%) \cdot (1+22.47\%) = 1.2247^2 \approx (1+50\%)$. If the 12-month interest rate is 213.8%, what is the 1-month interest rate?

$$(1+r)^{12} \approx 1 + 213.8\%$$
$$r = \sqrt[12]{1+213.8\%} - 1 = (1+213.8\%)^{1/12} - 1 \approx 10\%$$

➤ Exponentiation, Book Appendix, p. A-1

You can determine fractional time interest rates via compounding, too.

Interestingly, compounding works even over fractional time periods. Say the overall interest rate is 5% per year, and you want to find out what the rate of return over half a year would be. Because $(1+r_{0.5})^2 = (1+r_1)$, you would compute

$$(1+r_{0.5}) = (1+r_1)^{0.5} = (1+5\%)^{0.5} \approx 1 + 2.4695\% = 1.024695$$

ANECDOTE Life Expectancy and Credit

Your life expectancy may be 80 years, but 30-year bonds existed even in an era when life expectancy was only 25 years—at the time of Hammurabi, around 1700 B.C.E. (Hammurabi established the Kingdom of Babylon and is famous for the Hammurabi Code, the first known legal system.) Moreover, four thousand years ago, Mesopotamians already solved interesting financial problems. A cuneiform clay tablet contains the oldest known interest rate problem for prospective students of the financial arts. The student must figure out how long it takes for 1 mina of silver, growing at 20% interest per year, to reach 64 minae. Because the interest compounds in an odd way (20% of the principal is accumulated until the interest is equal to the principal, and then it is added back to the principal), the answer to this problem is 30 years, rather than 22.81 years. This is not an easy problem to solve—and it even requires knowledge of logarithms!

Check—compounding 2.4695% over two (6-month) periods indeed yields 5%:

$$(1 + 2.4695\%) \cdot (1 + 2.4695\%) = 1.024695^2 \approx (1 + 5\%)$$

$$(1 + r_{0.5}) \cdot (1 + r_{0.5}) = (1 + r_{0.5})^2 = (1 + r_1)$$

You need logs to determine the time needed to get x times your money.

If you know how to use logarithms, you can also determine with the same formula how long it will take at the current interest rate to double or triple your money. For example, at an interest rate of 3% per year, how long would it take you to double your money?

$$(1 + 3\%)^x = (1 + 100\%) \Leftrightarrow x = \frac{\log(1 + 100\%)}{\log(1 + 3\%)} = \frac{\log(2.00)}{\log(1.03)} \approx 23.5$$

$$(1 + r)^t = (1 + r_t) \Leftrightarrow t = \frac{\log(1 + r_t)}{\log(1 + r)}$$

How Bad Are Mistakes?

ADDING OR COMPOUNDING INTEREST RATES?

Adding rather than compounding can make forgivably small mistakes in certain situations—but don't be ignorant of what you are doing.

Unfortunately, when it comes to interest rates in the real world, many users are casual, sometimes to the point where they are outright wrong. Some people mistakenly add interest rates instead of compounding them. When the investments, the interest rates, and the time periods are small, the difference between the correct and incorrect computation can be minor, so this practice can be acceptable, even if it is wrong. For example, when interest rates are 10%, compounding yields

$$(1 + 10\%) \cdot (1 + 10\%) - 1 = 1.1^2 - 1 = 21\%$$

$$(1 + r) \cdot (1 + r) - 1 = r_2$$

$$= 1 + r + r + r \cdot r - 1$$

which is not exactly the same as the simple sum of two r's, which comes to 20%. The difference between 21% and 20% is the "cross-term" $r \cdot r$. This cross-product is especially unimportant if both rates of return are small. If the interest rate were both

1%, the cross-term would be 0.0001. This is indeed small enough to be ignored in most situations, and therefore a forgivable approximation. However, when you compound over many periods, you will accumulate more and more cross-terms, and eventually the quality of your approximation will deteriorate.

SOLVE NOW!

Q 2.10 If the 1-year rate of return is 20% and interest rates are constant, what is the 5-year holding rate of return?

Q 2.11 If you invest $2,000 today and it earns 25% per year, how much will you have in 15 years?

Q 2.12 What is the holding rate of return for a 20-year investment that earns 5%/year each year? What would a $200 investment grow to?

Q 2.13 A project lost one-third of its value each year for 5 years. What was its total holding rate of return? How much is left if the original investment was $20,000?

Q 2.14 If the 5-year holding rate of return is 100% and interest rates are constant, what is the (compounding) annual interest rate?

Q 2.15 What is the quarterly interest rate if the annual interest rate is 50%?

Q 2.16 If the per-year interest rate is 5%, what is the 2-year total interest rate?

Q 2.17 If the per-year interest rate is 5%, what is the 10-year total interest rate?

Q 2.18 If the per-year interest rate is 5%, what is the 100-year total interest rate? How does this compare to 100 times 5%?

Q 2.19 At a constant rate of return of 6% per annum, how many years does it take you to triple your money?

2.4C HOW BANKS QUOTE INTEREST RATES

Banks add to the confusion, quoting interest rates using strange but traditional conventions.

Banks and many other financial institutions use a number of conventions for quoting interest rates that may surprise you.

An **annual percentage yield** (**APY**) is the simple rate of return. (It is what our book calls an interest rate. Your bank sometimes calls this an **annual equivalent rate** (**AER**) or an **effective annual rate**.) If you invest $100, and the APY is 10%, you end up with $110 at the end of the year.

The **interest rate** stated without qualification is not really a rate of return, but just a method of quoting. The true daily interest rate is this annual interest quote divided by 365 (or 360 by another convention). For example, if your bank quotes you an annual interest rate of 10%, it means that the daily interest rate is $10\%/365 \approx 0.0274\%$. This is also why your bank may call this the **annual rate, compounded daily**. Therefore, if you leave your money in the bank for 1 year, you earn a true

$$\text{Actual Rate of Return} = [1 + (10\%/365)]^{365} - 1 \approx 10.52\%$$

In sum, at a quoted bank interest rate of 10%, $100 turns into $110.52 after 1 year.

An **annual percentage rate** (**APR**) is the rate that a bank is required to quote on loans it extends, according to the Consumer Credit Act of 1980. This act requires lenders to quote an "annual rate, compounded monthly," thus rendering APR as a number similar to a plain interest quote (not an APY). For example, if the quote to you is 10% per annum, then the lender will collect $(1 + 10\%/12)^{12} - 1 \approx 10.47\%$ per year on the money lent to you. For every \$100 you borrow, you will have to pay the bank \$10.47 every year. However, in contrast to the simple interest quote, APR not only has a different compounding interval, but is also required to reflect other closing costs and fees in order to aid consumers. Yet even though APR is supposedly a standardized measure, there are still enough variations in common use that comparing APRs may not always be comparing apples to apples.

A **certificate of deposit** (**CD**) is a longer-term investment vehicle than a savings account deposit. If your bank wants you to deposit your money in a CD, do you think it will put the more traditional interest rate quote or the APY on its sign in the window? Because the APY of 10.52% looks larger and thus more appealing to depositors than the traditional 10% interest rate quote, most banks advertise the APY for deposits. If you want to borrow money from your bank, do you think your loan agreement will similarly emphasize the APY? No. Most of the time, banks leave this number to the fine print and focus on the APR (or the traditional interest rate quote) instead.

Interest rates are not intrinsically difficult but they can be tedious, and definitional confusions abound in their world. My best advice when money is at stake: If in doubt, ask how the interest rate is computed! Even better, ask for a simple illustrative calculation.

SOLVE NOW!

Q 2.20 If you earn an (effective) interest rate of 12% per annum, how many basis points do you earn in interest on a typical calendar day?

Q 2.21 If the bank quotes an interest rate of 12% per annum (not as an effective interest rate), how many basis points do you earn in interest on a typical day?

Q 2.22 If the bank states an *effective* interest rate of 12% per annum, and there are 52.15 weeks, how much interest do you earn on a deposit of \$100,000 over 1 week?

Q 2.23 If the bank quotes interest of 12% per annum, and there are 52.15 weeks, how much interest do you earn on a deposit of \$100,000 over 1 week?

Q 2.24 If the bank quotes interest of 12% per annum, and there are 52.15 weeks, how much interest do you earn on a deposit of \$100,000 over 1 year?

Q 2.25 If the bank quotes an interest rate of 6% per annum, what does a deposit of \$100 in the bank come to after 1 year?

Q 2.26 If the bank quotes a loan APR rate of 8% per annum, compounded monthly, and there are no fees, what do you have to pay back in 1 year if you borrow \$100 from the bank?

2.5 PRESENT VALUES, DISCOUNTING, AND CAPITAL BUDGETING

Capital budgeting: Should you budget capital for a project?

Now turn to the flip side of the future value problem: If you know how much money you will have next year, what does this correspond to in value *today*? This is especially important in a corporate context, where the question is, "Given that Project X will return $1 million in 5 years, how much should you be willing to pay to undertake this project today?" The process entailed in answering this question is called **capital budgeting** and is at the heart of corporate decision making. (The origin of the term was the idea that firms have a "capital budget," and that they must allocate capital to their projects within that budget.)

➤ Formula 2.1, p. 16

The "present value formula" *is* nothing but the rate of return definition—inverted to translate future cash flows into (equivalent) today's dollars.

Start again with the rate of return formula

$$r_1 = \frac{C_1 - C_0}{C_0} = \frac{C_1}{C_0} - 1$$

You only need to turn this formula around to answer the following question: If you know the prevailing interest rate in the economy (r_1) and the project's future cash flows (C_1), what is the project's value to you *today*? In other words, you are looking for the **present value** (**PV**)—the amount a future sum of money is worth today, given a specific rate of return. For example, if the interest rate is 10%, how much would you have to save (invest) to receive $100 next year? Or, equivalently, if your project will return $100 next year, what is the project worth to you today? The answer lies in the present value formula, which translates future money into today's money. You merely need to rearrange the rate of return formula to solve for the present value:

$$C_0 = \frac{\$100}{1 + 10\%} = \frac{\$100}{1.1} \approx \$90.91$$

$$C_0 = \frac{C_1}{1 + r_1} = \text{PV}(C_1)$$

Check this—investing $90.91 at an interest rate of 10% will indeed return $100 next period:

$$10\% \approx \frac{\$100 - \$90.91}{\$90.91} = \frac{\$100}{\$90.91} - 1 \Leftrightarrow (1 + 10\%) \cdot \$90.91 \approx \$100$$

$$r_1 = \frac{C_1 - C_0}{C_0} = \frac{C_1}{C_0} - 1 \Leftrightarrow (1 + r_1) \cdot C_0 = C_1$$

Discounting translates future cash into today's equivalent.

This is the **present value formula**, which uses a division operation known as **discounting**. (The term "discounting" indicates that we are reducing a value, which is exactly what we are doing when we translate future cash into current cash.) If you wish, you can think of discounting—the conversion of a future cash flow amount into its equivalent present value amount—as the *reverse* of compounding.

Present value varies inversely with the cost of capital.

Thus, the present value (PV) of next year's $100 is $90.91—the value today of future cash flows. Let's say that this $90.91 is what the project costs. If you can borrow or lend at the interest rate of 10% elsewhere, then you will be indifferent between receiving \$100 next year and receiving \$90.91 for your project today. In contrast, if the standard rate of return in the economy were 12%, your specific project would not be a good deal. The project's present value would be

$$\text{PV}(C_1) = \frac{\$100}{1 + 12\%} = \frac{\$100}{1.12} \approx \$89.29$$
$$C_0 = \frac{C_1}{1 + r_1} = \text{PV}(C_1)$$

which would be less than its cost of \$90.91. But if the standard economy-wide rate of return were 8%, the project would be a great deal. Today's present value of the project's future payoff would be

$$\text{PV}(C_1) = \frac{\$100}{1 + 8\%} = \frac{\$100}{1.08} \approx \$92.59$$
$$C_0 = \frac{C_1}{1 + r_1} = \text{PV}(C_1)$$

which would exceed the project's cost of \$90.91. It is the present value of the project, weighed against its cost, that should determine whether you should undertake a project today or avoid it. The present value is also the answer to the question, "How much would you have to save at current interest rates today if you wanted to have a specific amount of money next year?"

The PV formula with 2 periods.

Let's extend the time frame in our example. If the interest rate were 10% per period, what would \$100 in 2 periods be worth today? The value of the \$100 is then

$$\text{PV}(C_2) = \frac{\$100}{(1 + 10\%)^2} = \frac{\$100}{1.21} \approx \$82.64$$
$$\text{PV}(C_2) = \frac{C_2}{(1 + r)^2} = C_0 \qquad (2.2)$$

Note the 21%. In 2 periods, you could earn a rate of return of $(1 + 10\%) \cdot (1 + 10\%) - 1 = 1.1^2 - 1 = 21\%$ elsewhere, so this is your appropriate comparable rate of return.

The interest rate can be called the "cost of capital."

This discount rate—the rate of return, r, with which the project can be financed—is often called the **cost of capital**. It is the rate of return at which you can raise money elsewhere. In a perfect market, this cost of capital is also the **opportunity cost** that you bear if you fund your specific investment project instead of the alternative next-best investment elsewhere. Remember—you can invest your money at this rate in another project instead of investing it in this one. The better these alternative projects in the economy are, the higher will be your cost of capital, and the lower will be the value of your specific investment project with its specific cash flows. An investment that promises \$1,000 next year is worth less today if you can earn 50% rather than 5% elsewhere. A good rule is to always add mentally the word "opportunity" before "cost of capital"—it is always your **opportunity cost of capital**. (In this part of our book, I will just tell you what the economy-wide rate of return is—here 10%—

for borrowing or investing. In later chapters, you will learn how this rate of return is determined.)

IMPORTANT: Always think of the r in the present value denominator as your "opportunity" cost of capital. If you have great opportunities elsewhere, your projects have to be discounted at high discount rates. The discount rate, the cost of capital, and the required rate of return are really all just names for the same factor.

The discount factor is a simple function of the cost of capital.

When you multiply a future cash flow by its appropriate **discount factor**, you end up with its present value. Looking at Formula 2.2, you can see that this discount factor is the quantity

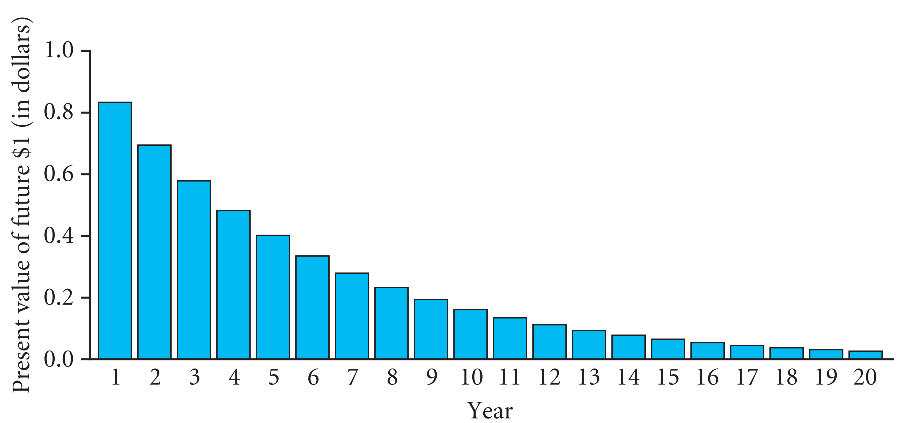
$$\left(\frac{1}{1+21\%}\right) \approx 0.8264$$

$$\left(\frac{1}{1+r_2}\right)$$

In other words, the discount factor translates 1 dollar in the future into the equivalent amount of dollars today. In the example, at a 2-year 21% rate of return, a dollar in 2 years is worth about 83 cents today. Because interest rates are usually positive, discount factors are usually less than 1—a dollar in the future is worth less than a dollar today. (Sometimes, and less correctly, people call this the **discount rate**, but this name should be used for r_t instead.)

The discount rate is higher for years farther out, so the discount factor is lower.

Figure 2.2 shows how the discount factor declines when the cost of capital is 20% per annum. After about a decade, any dollar the project earns is worth less than 20 cents to you today. If you compare Figure 2.1 to Figure 2.2, you should notice how each is the "flip side" of the other.



Each bar is $1/(1+20\%) \approx 83.3\%$ of the size of the bar to its left. After 20 years, the last bar is 0.026 in height. This means that $1 in 20 years is worth 2.6 cents in money today.

FIGURE 2.2 Discounting over 20 Years at a Cost of Capital of 20% per Annum

IMPORTANT: The cornerstones of finance are the following formulas:

$$\text{Rate of Return:} \quad r_t = \frac{C_t - C_0}{C_0} = \frac{C_t}{C_0} - 1$$

Rearrange the formula to obtain the future value:

$$\text{Future Value:} \quad \text{FV}_t = C_t = C_0 \cdot (1 + r_t) = C_0 \cdot (1 + r)^t$$

The process of obtaining r_t is called compounding, and it works through the "one-plus" formula:

$$\text{Compounding:} \quad \underbrace{(1 + r_t)}_{\text{Total Holding Rate of Return}} = \underbrace{(1 + r)}_{\text{First Period Rate of Return}} \cdot \underbrace{(1 + r)}_{\text{Second Period Rate of Return}} \cdots \underbrace{(1 + r)}_{\text{Third Period Rate of Return}}$$

Rearrange the formula again to obtain the present value:

$$\text{Present Value:} \quad \text{PV} = C_0 = \frac{C_t}{(1 + r_t)} = \frac{C_t}{(1 + r)^t}$$

The process of translating C_t into C_0—that is, the multiplication of a future cash flow by $1/(1 + r_t)$—is called discounting. The discount factor is:

$$\text{Discount Factor:} \quad \frac{1}{(1 + r_t)} = \frac{1}{(1 + r)^t}$$

It translates 1 dollar at time t into its equivalent value today.

Bonds' present values and the prevailing interest rates move in opposite directions.

Remember how bonds are different from savings accounts? The former is pinned down by its promised fixed future payments, while the latter pays whatever the daily interest rate is. This induces an important relationship between the value of bonds and the prevailing interest rates—*they move in opposite directions*. For example, if you have a bond that promises to pay \$1,000 in 1 year, and the prevailing interest rate is 5%, the bond has a present value of $\$1{,}000/1.05 \approx \$952.38$. If the prevailing interest rate suddenly increases to 6% (and thereby becomes your new opportunity cost of capital), the bond's present value becomes $\$1{,}000/1.06 \approx \$943.40$. You would have lost \$8.98, which is about 0.9% of your original \$952.38 investment. The value of your fixed-bond payment in the future has gone down, because investors can now do better than your 5% by buying new bonds. They have better opportunities elsewhere in the economy. They can earn a rate of return of 6%, not just 5%, so if you wanted to sell your bond now, you would have to sell it at a discount to leave the next buyer a rate of return of 6%. If you had delayed your investment, the sudden change to 6% would have done nothing to your investment. On the other hand, if the prevailing interest rate suddenly drops to 4%, then your bond will be more valuable. Investors would be willing to pay $\$1{,}000/1.04 \approx \$961.54$, which is an immediate \$9.16 gain. The inverse relationship between prevailing interest rates and bond prices is general and worth noting.

IMPORTANT: The price and the implied rate of return on a bond with fixed payments move in opposite directions. When the price of the bond goes up, its implied rate of return goes down. When the price of the bond goes down, its implied rate of return goes up.

SOLVE NOW!

Q 2.27 A project has a cost of capital of 30%. The final payoff is \$250. What should it cost today?

Q 2.28 A bond promises to pay \$150 in 12 months. The annual true interest rate is 5% per annum. What is the bond's price today?

Q 2.29 A bond promises to pay \$150 in 12 months. The bank quotes you interest of 5% per annum, compounded daily. What is the bond's price today?

Q 2.30 If the cost of capital is 5% per annum, what is the discount factor for a cash flow in 2 years?

Q 2.31 Interpret the meaning of the discount factor.

Q 2.32 What are the units on rates of return, discount factors, future values, and present values?

Q 2.33 Would it be good or bad for you, in terms of the present value of your liabilities, if your opportunity cost of capital increased?

Q 2.34 The price of a bond that offers a safe promise of \$100 in 1 year is \$95. What is the implied interest rate? If the bond's interest rate suddenly jumped up by 150 basis points, what would the bond price be? How much would an investor gain/lose if she held the bond while the interest rate jumped up by these 150 basis points?

2.6 NET PRESENT VALUE

Present values are alike and thus can be added, subtracted, compared, and so on.

An important advantage of present value is that all cash flows are translated into the same unit: cash today. To see this, say that a project generates \$10 in 1 year and \$8 in 5 years. You cannot add up these different future values to come up with \$18—it would be like adding apples and oranges. However, if you translate both future cash flows into their present values, you *can* add them. For example, if the interest rate was 5% per annum (so $(1 + 5\%)^5 = (1 + 27.6\%)$ over 5 years), the present value of these two cash flows together would be

$$\text{PV}(\$10 \text{ in 1 year}) = \frac{\$10}{1.05} \approx \$9.52$$

$$\text{PV}(\$8 \text{ in 5 years}) = \frac{\$8}{1.05^5} \approx \$6.27$$

$$\text{PV}(C_t) = \frac{C_t}{(1+r)^t}$$

Therefore, the total value of the project's future cash flows *today* (at time 0) is \$15.79.

The definition and use of NPV.

The **net present value** (**NPV**) of an investment is the present value of all its future cash flows minus the present value of its cost. It is really the same as present value, except that the word "net" up front reminds you to add and subtract *all* cash flows, including the *upfront* investment outlay today. The NPV calculation method is always the same:

1. Translate all future cash flows into today's dollars.
2. Add them all up. This is the present value of all future cash flows.
3. Subtract the initial investment.

NPV is the most important method for determining the value of projects. It is a cornerstone of finance. Let's assume that you have to pay \$12 to buy this particular project with its \$10 and \$8 cash flows. In this case, it is a positive NPV project, because

$$\text{NPV} = -\$12 + \frac{\$10}{1.05} + \frac{\$8}{1.05^5} \approx \$3.79$$

$$C_0 + \frac{C_1}{1+r_1} + \frac{C_5}{(1+r)^5} = \text{NPV}$$

(For convenience, we omit the 0 subscript for NPV, just as we did for PV.)

Think about what NPV means, and how it can be justified.

There are a number of ways to understand net present value.

- One way is to think of the NPV of \$3.79 as the difference between the market value of the future cash flows (\$15.79) and the project's cost (\$12)—this difference is the "value added."
- Another way to think of your project is to compare its cash flows to an equivalent set of bonds that exactly *replicates* them. In this instance, you would want to purchase a 1-year bond that promises \$10 next year. If you save \$9.52—at a 5% interest rate—you will receive \$10. Similarly, you could buy a 5-year bond that promises \$8 in year 5 for \$6.27. Together, these two bonds exactly replicate the project cash flows. The **law of one price** tells you that your project should be worth as much as this bond project—the cash flows are identical. You would have had to put away \$15.79 today to buy these bonds, but your project can deliver these cash flows at a cost of only \$12—much cheaper and thus better than your bond alternative.

Yet another way to justify NPV: opportunity cost.

- There is yet another way to think of NPV. It tells you how your project compares to the alternative opportunity of investing in the capital markets. These opportunities are expressed in the denominator through the discount factor. What would you get if you took your \$12 and invested it in the capital markets instead of in your project? Using the future value formula, you know that you could earn a 5% rate of return from now to next year, and 27.6% from now to 5 years. Your \$12 would grow into \$12.60 by next year. You could take out the same \$10 cash flow that your project gives you and be left with \$2.60 for reinvestment. Over the next 4 years, at the 5% interest rate, this \$2.60 would grow into \$3.16. But your project would do better for you, giving you \$8. Thus, your project achieves a higher rate of return than the capital markets alternative would achieve.

The conclusion of this argument is not only the simplest but also the best capital budgeting rule: If the NPV is positive, as it is here, you should take the project. If it is negative, you should reject the project. If it is zero, it does not matter.

The correct capital budgeting rule: Take all positive NPV projects.

IMPORTANT:

- The **net present value formula** is

$$\begin{aligned} \text{NPV} &= C_0 + \text{PV}(C_1) + \text{PV}(C_2) + \text{PV}(C_3) + \text{PV}(C_4) + \cdots \\ &= C_0 + \frac{C_1}{1+r_1} + \frac{C_2}{1+r_2} + \frac{C_3}{1+r_3} + \frac{C_4}{1+r_4} + \cdots \\ &= C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \frac{C_4}{(1+r)^4} + \cdots \end{aligned}$$

  The subscripts are time indexes, C_t is the net cash flow at time t (positive for inflows, negative for outflows), and r_t is the relevant interest rate for investments from now to time t. With constant interest rates, $r_t = (1+r)^t - 1$.
- The **net present value capital budgeting rule** states that you should accept projects with a positive NPV and reject projects with a negative NPV.
- Taking positive NPV projects increases the value of the firm. Taking negative NPV projects decreases the value of the firm.
- NPV is definitively the best method for capital budgeting—the process by which you should accept or reject projects.

The NPV formula is so important that you must memorize it.

Let's work another NPV example. A project costs \$900 today, yields \$200/year for 2 years, then \$400/year for 2 years, and finally requires a cleanup expense of \$100. The prevailing interest rate is 5% per annum. These cash flows are summarized in Table 2.1. Should you take this project?

Let's work a project NPV example.

1. You need to determine the cost of capital for tying up money for 1 year, 2 years, 3 years, and so on. The compounding formula is

$$(1 + r_t) = (1 + r)^t = (1.05)^t = 1.05^t$$

   So for money right now, the cost of capital r_0 is $1.05^0 - 1 = 0$; for money in 1 year, r_1 is $1.05^1 - 1 = 5\%$; for money in 2 years, r_2 is $1.05^2 - 1 = 10.25\%$. And so on.

First, determine your multiyear costs of capital.

2. You need to translate the cost of capital into discount factors. Recall that these are 1 divided by 1 plus your cost of capital. A dollar in 1 year is worth $1/(1 + 5\%) = 1/1.05 \approx 0.9524$ dollars today. A dollar in 2 years is worth $1/(1 + 5\%)^2 = 1/1.05^2 \approx 0.9070$. And so on.
3. You can now translate the future cash flows into their present value equivalents by multiplying the payoffs by their appropriate discount factors. For example, the \$200 cash flow at time 1 is worth about $0.9524 \cdot \$200 \approx \$190.48$.

TABLE 2.1 HYPOTHETICAL PROJECT CASH FLOW TABLE

Time	Project Cash Flow	Interest Rate Annualized	Interest Rate Holding	Discount Factor	Present Value
t	C_t	r	r_t	$\frac{1}{(1+r)^t}$	$PV(C_t)$
Today	−$900	5.00%	0.00%	1.000	−$900.00
Year +1	+$200	5.00%	5.00%	0.9524	+$190.48
Year +2	+$200	5.00%	10.25%	0.9070	+$181.41
Year +3	+$400	5.00%	15.76%	0.8638	+$345.54
Year +4	+$400	5.00%	21.55%	0.8227	+$329.08
Year +5	−$100	5.00%	27.63%	0.7835	−$78.35
				Net Present Value (Sum):	$68.16

As a manager, you must estimate your project cash flows. The appropriate interest rate (also called cost of capital in this context) is provided to you by the opportunity cost of your investors—determined by the supply and demand for capital in the broader economy, where your investors can place their capital instead. The "Project Cash Flow" and the left interest rate column are the two input columns. The remaining columns are computed from these inputs. The goal is to calculate the final column.

4. Because present values are additive, you then sum up all the terms to compute the overall net present value. Make sure you include the original upfront cost as a negative.

Consequently, the project NPV is $68.16. Because this is a positive value, you should take this project.

If the upfront cost was higher, you should not take the project.

However, if the upfront expense was $1,000 instead of $900, the NPV would be negative (−$31.84), and you would be better off investing the money into the appropriate sequence of bonds from which the discount factors were computed. In this case, you should have rejected the project.

SOLVE NOW!

Q 2.35 Work out the present value of your tuition payments for the next 2 years. Assume that the tuition is $30,000 per year, payable at the start of the year. Your first tuition payment will occur in 6 months, and your second tuition payment will occur in 18 months. You can borrow capital at an interest rate of 6% per annum.

Q 2.36 Write down the NPV formula from memory.

Q 2.37 What is the NPV capital budgeting rule?

Q 2.38 Determine the NPV of the project in Table 2.1, if the per-period interest rate were 8% per year, not 5%. Should you take this project?

Q 2.39 You are considering a 3-year lease for a building, where you have to make one payment now, one in a year, and a final one in 2 years.

(a) Would you rather pay $1,000,000 up front, then $500,000 each in the following two years; or would you rather pay $700,000 each year?

(b) If the interest rate is 10%, what equal payment amount (rather than $700,000) would leave you indifferent? (This is also called the equivalent annual cost (EAC).)

➤ Section 3.4, "Projects With Different Lives and Rental Equivalents," p. 60

Q 2.40 Use a spreadsheet to answer the following question: Car dealer A offers a car for $2,200 up front (first payment), followed by $200 lease payments over the next 23 months. Car dealer B offers the same lease at a flat $300 per month (i.e., your first upfront payment is $300). Which lease do you prefer if the interest rate is 0.5% per month?

2.6A APPLICATION: ARE FASTER-GROWING FIRMS BETTER BARGAINS?

The firm's price should incorporate the firm's attributes.

Let's work another NPV problem applied to companies overall. Would it make more sense to invest in companies that grow quickly rather than slowly? If you wish, you can think of this question loosely as asking whether you should buy stocks in a fast-growing company like Google or in a slow-growing company like Procter & Gamble. The answer will be that this choice does not matter in a perfect market. Whether a company is growing quickly or slowly is already incorporated in the firm's price today, which is just the present value of the firm's cash flows that will accrue to the owners. Therefore, neither is the better deal.

Should you invest in a fast-grower or a slow-grower?

For example, consider company "Grow" (G) that will produce over the next 3 years

$$G_1 = \$100 \qquad G_2 = \$150 \qquad G_3 = \$250$$

and company "Shrink" (S) that will produce

$$S_1 = \$100 \qquad S_2 = \$90 \qquad S_3 = \$80$$

Is G not a better company to buy than S?

Let's find out: Compute the values.

There is no uncertainty involved, and both firms face the same cost of capital of 10% per annum. The price of G today is

$$\text{PV}(G) = \frac{\$100}{1.1^1} + \frac{\$150}{1.1^2} + \frac{\$250}{1.1^3} \approx \$402.70 \qquad (2.3)$$

and the price of S today is

$$\text{PV}(S) = \frac{\$100}{1.1^1} + \frac{\$90}{1.1^2} + \frac{\$80}{1.1^3} \approx \$225.39$$

Your investment dollar grows at the same 10% rate. Your investment's growth rate is disconnected from the cash flow growth rate.

If you invest in G, then next year you will have $100 cash and own a company with $150 and $250 cash flows coming up. G's value at time 1 (so PV now has subscript 1) will thus be

$$\text{PV}_1(G) = \$100 + \frac{\$150}{1.1^1} + \frac{\$250}{1.1^2} \approx \$442.98$$

Your investment will have earned a rate of return of $\$442.98/\$402.70 - 1 \approx 10\%$. If you instead invest in S, then next year you will receive $100 cash and own a company with "only" \$90 and \$80 cash flows coming up. *S*'s value will thus be

$$\text{PV}_1(S) = \$100 + \frac{\$90}{1.1^1} + \frac{\$80}{1.1^2} \approx \$247.93$$

Your investment will have earned a rate of return of \$247.93/\$225.39 − 1 ≈ 10%. In either case, you will earn the fair rate of return of 10%. Whether cash flows are growing at a rate of +50%, −10%, +237.5%, or −92% is irrelevant: *The firms' market prices today already reflect their future growth rates*. There is no necessary connection between the growth rate of the underlying project cash flows or earnings and the growth rate of your investment money (i.e., your expected rate of return).

Any sudden wealth gains would accrue to existing shareholders, not to new investors.

Make sure you understand the thought experiment here: This statement that higher-growth firms do not necessarily earn a higher rate of return does not mean that a firm in which managers succeed in increasing the future cash flows at no extra investment cost will not be worth more. Such firms will indeed be worth more, and the current owners will benefit from the rise in future cash flows, but this will also be reflected immediately in the price at which you can purchase this firm. This is an important corollary worth repeating. If General Electric has just won a large defense contract (like the equivalent of a lottery), shouldn't you purchase GE stock to participate in the windfall? Or if Wal-Mart managers do a great job and have put together a great firm, shouldn't you purchase Wal-Mart stock to participate in this windfall? The answer is that you cannot. The old shareholders of Wal-Mart are no dummies. They know the capabilities of Wal-Mart and how it will translate into cash flows. Why should they give you, a potential new shareholder, a special bargain for something to which you contributed nothing? Just providing more investment funds is not a big contribution—after all, there are millions of other investors equally willing to provide funds at the appropriately higher price. It is competition—among investors for providing funds and among firms for obtaining funds—that determines the expected rate of return that investors receive and the cost of capital that firms pay. There is actually a more general lesson here. Economics tells you that you must have a scarce resource if you want to earn above-normal profits. Whatever is abundant and/or provided by many will not be tremendously profitable.

SOLVE NOW!

Q 2.41 Assume that company *G* pays no interim dividends, so you receive \$536 at the end of the project. What is *G*'s market value at time 1, 2, and 3? What is your rate of return in each year? Assume that the cost of capital is still 10%.

Q 2.42 Assume that company *G* pays out the full cash flows (refer to the text example) in earnings each period. What is *G*'s market value at time 1, 2, and 3? What is your rate of return in each year?

Q 2.43 One month ago, a firm suffered a large court award against it that will force it to pay compensatory damages of \$100 million next January 1. Are shares in this firm a bad buy until January 2?

SUMMARY

This chapter covered the following major points:

- A perfect market assumes no taxes, no transaction costs, no opinion differences, and the presence of many buyers and sellers.
- A bond is a claim that promises to pay an amount of money in the future. Buying a bond is extending a loan. Issuing a bond is borrowing. Bond values are determined by their future payoffs.
- One hundred basis points are equal to 1%.
- The time value of money means that 1 dollar today is worth more than 1 dollar tomorrow, because of the interest that it can earn.
- Returns must not be averaged, but compounded over time.
- Interest rate quotes are *not* interest rates. For example, stated annual rates are usually not the effective annual rates that your money will earn in the bank. If in doubt, ask!
- The discounted present value (PV) translates future cash values into present cash values. The net present value (NPV) is the sum of all present values of a project, including the investment cost (usually, a negative upfront cash flow today).
- The values of bonds and interest rates move in opposite directions. A sudden increase in the prevailing economy-wide interest rate decreases the present value of a bond's future payouts and therefore decreases today's price of the bond. Conversely, a sudden decrease in the prevailing economy-wide interest rate increases the present value of a bond's future payouts and therefore increases today's price of the bond.
- The NPV formula can be written as

$$\begin{aligned} \text{NPV} &= C_0 + \frac{C_1}{1+r_1} + \frac{C_2}{1+r_2} + \cdots \\ &= C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots \end{aligned}$$

  In this context, r is called the discount rate or cost of capital, and $1/(1+r)$ is called the discount factor.
- The net present value capital budgeting rule states that you should accept projects with a positive NPV and reject projects with a negative NPV.
- In a perfect market, firms are worth the present value of their assets. Whether firms grow quickly or slowly does not make them more or less attractive investments in a perfect market because their prices always already reflect the present value of future cash flows.
- In a perfect market, the gains from sudden surprises accrue to old owners, not new capital providers, because old owners have no reason to want to share the spoils.

KEY TERMS

AER, 23
annual equivalent rate, 23
annual percentage rate, 24
annual percentage yield, 23
annual rate, compounded daily, 23
APR, 24
APY, 23
basis point, 17
bond, 14
capital budgeting, 25
capital gain, 16
capital loss, 17
CD, 24
certificate of deposit, 24
compounding, 20
compound interest, 19
cost of capital, 26
coupon, 16
coupon yield, 16
current yield, 16
discount factor, 27
discounting, 25
discount rate, 27
dividend, 16
dividend yield, 16
effective annual rate, 23
fixed income, 14
future value, 18
FV, 18
holding rate of return, 20
interest, 14
interest rate, 14, 23
law of one price, 30
loan, 14
maturity, 14
net present value, 30
net present value capital budgeting rule, 31
net present value formula, 31
net return, 15
NPV, 30
opportunity cost, 26
opportunity cost of capital, 26
perfect market, 13
present value, 25
present value formula, 25
PV, 25
rate of return, 16
rental yield, 16
return, 15
time value of money, 18

SOLVE NOW! SOLUTIONS

Q 2.1 The four perfect market assumptions are no taxes, no transaction costs, no differences in opinions, and no large buyers or sellers.

Q 2.2 A savings deposit is an investment in a series of short-term bonds.

Q 2.3 $r = (\$1{,}050 - \$1{,}000)/\$1{,}000 = 5\%$

Q 2.4 $r = \frac{\$25}{\$1,000} = 2.5\%$

Q 2.5 Yes, $10 = 1{,}000\%$.

Q 2.6 The dividend yield would be $\$1/\$40 = 2.5\%$, the capital gain would be $\$45 - \$40 = \$5$, so that its capital gain yield would be $\$5/\$40 = 12.5\%$, and the total rate of return would be $(\$46 - \$40)/\$40 = 15\%$.

Q 2.7 $1\% = 100$ basis points, so an increase of 3% is 300 basis points.

Q 2.8 20 basis points are 0.2%, so the interest rate declined from 10.0% to 9.8%.

Q 2.9 $r = 30\% = (x - \$250)/\$250 \quad\quad \Longrightarrow x = 1.3 \cdot \$250 = \$325$

Q 2.10 $1.20^5 - 1 \approx 148.83\%$

Q 2.11 $\$2{,}000 \cdot 1.25^{15} \approx \$56{,}843.42$

Q 2.12 The total holding rate of return is $1.05^{20} - 1 \approx 165.33\%$, so you would end up with $\$200 \cdot (1 + 165.33\%) \approx \$530.66$.

Q 2.13 Losing one-third is a rate of return of −33%. To find the holding rate of return, compute $[1 + (-1/3)]^5 - 1 \approx -86.83\%$. About $(1 - 86.83\%) \cdot \$20{,}000 \approx \$2{,}633.74$ remains.

Q 2.14 $(1 + 100\%)^{1/5} - 1 \approx 14.87\%$

Q 2.15 $(1 + r_{0.25})^4 = (1 + r_1)$. Thus, $r_{0.25} = \sqrt[4]{1 + r_1} - 1 = 1.5^{1/4} - 1 \approx 10.67\%$.

Q 2.16 $r_2 = (1 + r_{0,1}) \cdot (1 + r_{1,2}) - 1 = 1.05 \cdot 1.05 - 1 = 10.25\%$

Q 2.17 $r_{10} = (1 + r_1)^{10} - 1 = 1.05^{10} - 1 \approx 62.89\%$

Q 2.18 $r_{100} = (1 + r_1)^{100} - 1 = 1.05^{100} - 1 = 130.5 \approx 13{,}050\%$. In words, this is about 130 times the initial investment, and substantially more than 500% (5 times the initial investment).

Q 2.19 Tripling is equivalent to earning a rate of return of 200%. Therefore, solve $(1 + 6\%)^x = (1 + 200\%)$, or $x \cdot \log(1.06) = \log(3.00)$ or $x = \log(3.00)/\log(1.06) \approx 18.85$ years.

Q 2.20 $(1 + r)^{365} = 1.12$. Therefore, $1.12^{(1/365)} - 1 \approx 0.00031054 = 0.031054\% \approx 3.1$ bp/day.

Q 2.21 The bank means to collect $12\%/365 \approx 3.288$ bp/day.

Q 2.22 The true daily interest rate, assuming 365 days, is $1.12^{1/365} - 1 \approx 0.031054\%$. To compute your true rate of return, compound this over 7 days: $(1 + 0.03105\%)^7 = 1.00031054^7 \approx 1.0021758$. (You could also compute the rate of return differently: There are 52.15 weeks in 365 days. Therefore, $r = (1 + 12\%)^{(1/52.15)} - 1 \approx 1.0021758$.) Your \$100,000 will grow into \$100,217.58. You would have earned \$217.58 in interest.

Q 2.23 With 12% in nominal APR interest *quoted*, you earn $12\%/365 \approx 0.032877\%$ per day. Therefore, the weekly rate of return is $(1 + 0.032877\%)^7 - 1 \approx 0.23036\%$. Your \$100,000 will grow into \$100,230.36. Note that you end up with more money from the 12% quoted rate than from the 12% effective rate.

Q 2.24 With 12% in nominal APR interest *quoted*, you earn $12\%/365 \approx 0.032877\%$ per day. Therefore, the annual rate of return is $(1 + 0.032877\%)^{365} - 1 \approx 12.747462\%$. Your \$100,000 will grow into \$112,747.46.

Q 2.25 The bank quote of 6% means that it will pay an interest rate of $6\%/365 \approx 0.0164384\%$ per day. This earns an actual interest rate of $(1 + 0.0164384\%)^{365} - 1 \approx 6.18\%$ per annum. Therefore, each invested \$100 grows to \$106.18, thus earning \$6.18 over the year.

Q 2.26 The bank quote of 8% means that you will have to pay an interest rate of $8\%/12 \approx 0.667\%$ per month. This earns an actual interest rate of $(1 + 0.667\%)^{12} - 1 \approx 8.30\%$ per annum. You will have to pay \$108.30 in repayment for every \$100 you borrowed.

Q 2.27 $r = 30\% = (\$250 - x)/x$. Thus, $x = \$250/1.30 \approx \$192.31$.

Q 2.28 $\$150/(1.05) \approx \$142.86$

Q 2.29 $\$150/[1 + (5\%/365)]^{365} \approx \$142.68$

Q 2.30 $1/[(1.05) \cdot (1.05)] \approx 0.9070$

Q 2.31 It is today's value in dollars for 1 future dollar, that is, at a specific point in time in the future.

Q 2.32 The rate of return and additional factors are unit-less. The latter two are in dollars (though the former is dollars in the future, while the latter is dollars today).

Q 2.33 Good. Your future payments would be worth less in today's money.

Q 2.34 The original interest rate is $\$100/\$95 - 1 \approx 5.26\%$. Increasing the interest rate by 150 basis points is 6.76%. This means that the price should be $\$100/(1.0676) \approx \$93.67$. A price change from \$95 to \$93.67 is a rate of return of $\$93.67/\$95 - 1 \approx -1.40\%$.

Q 2.35 The first tuition payment is worth $\$30{,}000/(1.06)^{1/2} \approx \$29{,}139$. The second tuition payment is worth $\$30{,}000/(1.06)^{3/2} \approx \$27{,}489$. Thus, the total present value is \$56,628.

Q 2.36 If you cannot write down the NPV formula by heart, do not go on until you have it memorized.

Q 2.37 Accept if NPV is positive. Reject if NPV is negative.

Q 2.38 $-\$900 + \$200/(1.08)^1 + \$200/(1.08)^2 + \$400/(1.08)^3 + \$400/(1.08)^4 - \$100/(1.08)^5 \approx \$0.14$. The NPV is positive. Therefore this is a worthwhile project that you should accept.

Q 2.39 For the 3-year building leases:

(a) Your preference depends on the interest rate. If the interest rate is zero, then you would prefer the \$2 million sum-total payment to the \$2.1 million rent. If the prevailing interest rate is less than 21.5%, it is better to lease. If it is more than 21.5%, you prefer the rent. For example, if it is 40%, the net present cost of the lease is \$1.612 million, while the net present cost of the rent is \$1.557 million.

(b) At a 10% interest rate, the total net present cost of the lease is $\$1 + \$0.5/1.1 + \$0.5/1.1^2 \approx \$1.868$ million. An equivalent rent contract must solve

$$x + \frac{x}{1.1} + \frac{x}{1.1^2} = \$1.868$$

Multiply by $1.1^2 = 1.21$

$$1.21 \cdot x + 1.1 \cdot x + x = \$1.868 \cdot 1.21$$

$$\Leftrightarrow x \cdot (1.21 + 1.1 + 1) = \$2{,}260.28$$

Therefore, the equivalent rental cost would be $x \approx \$682.864$.

Q 2.40 Lease A has an NPV of −\$6,535. Lease B has an NPV of −\$6,803. Therefore, lease A is cheaper.

Q 2.41 For easier naming, call 2000 your year 0. The firm's present value in 2000 is $\$536/1.10^3 \approx \$402.70$—but you already knew this. If you purchase this company, its value in 2001 depends on a cash flow stream that is \$0 in 2001, \$0 in year 2002, and \$536 in year 2003. It will be worth $\$536/1.10^2 \approx \$442.98$ in 2001. In 2002, your firm will be worth $\$536/1.10 \approx \$487.27$. Finally, in 2003, it will be worth \$536. Each year, you expect to earn 10%, which you can compute from the four firm values.

Q 2.42 Again, call 2000 your year 0. The firm's present value in 2000 is based on dividends of \$100, \$150, and \$250 in the next three years. The firm value in 2000 is the \$402.70 from page 33. The firm value in 2001 was also worked out to be \$442.98, but you immediately receive \$100 in cash, so the firm is worth only $\$442.98 - \$100 = \$342.98$. As an investor, you would have earned a rate of return of $\$442.98/\$402.70 - 1 \approx 10\%$. The firm value in 2002 is $\text{PV}_2(G) = \$250/1.1 \approx \$227.27$, but you will also receive \$150 in cash, for a total firm-related wealth of \$377.27. In addition, you will have the \$100 from 2001, which would have grown to \$110—for a total wealth of \$487.27. Thus, starting with wealth of \$442.98 and ending up with wealth of \$487.27, you would have earned a rate of return of $\$487.27/\$442.98 - 1 \approx 10\%$. A similar computation shows that you will earn 10% from 2002 (\$487.27) to 2003 (\$536.00).

Q 2.43 No! The market will already have adjusted the price.

PROBLEMS

The ▭ *indicates problems available in* myfinancelab

Q 2.44 What is a perfect market? What were the assumptions made in this chapter that were not part of the perfect market scenario?

Q 2.45 What is the difference between a bond and a loan?

Q 2.46 In the text, I assumed you received the dividend at the end of the period. In the real world, if you received the dividend at the beginning of the period instead of the end of the period, could this change your effective rate of return? Why?

Q 2.47 Your stock costs \$100 today, pays \$5 in dividends at the end of the period, and then sells for \$98. What is your rate of return?

Q 2.48 The interest rate has just increased from 6% to 8%. How many basis points is this?

Q 2.49 Assume an interest rate of 10% per year. How much would you lose over 5 years if you had to give up interest on the interest—that is, if you received 50% instead of compounded interest?

Q 2.50 Over 20 years, would you prefer 10% per annum, with interest compounding, or 15% per annum but without interest compounding? (That is, you receive the interest, but it is put into an account that earns no interest, which is what we call simple interest.)

Q 2.51 A project returned +30%, then −30%. Thus, its arithmetic average rate of return was 0%. If you invested $25,000, how much did you end up with? Is your rate of return positive or negative? How would your overall rate of return have been different if you first earned –30% and then +30%?

Q 2.52 A project returned +50%, then −40%. Thus, its arithmetic average rate of return was +5%. Is your rate of return positive or negative?

Q 2.53 An investment for $50,000 earns a rate of return of 1% in each month of a full year. How much money will you have at year's end?

Q 2.54 There is always disagreement about what stocks are good purchases. The typical degree of disagreement is whether a particular stock is likely to offer, say, a 10% (pessimistic) or a 20% (optimistic) annualized rate of return. For a $30 stock today, what does the difference in belief between these two opinions mean for the expected stock price from today to tomorrow? (Assume that there are 365 days in the year. Reflect on your answer for a moment, and recognize that a $30 stock typically moves about ±$1 on a typical day. This unexplainable up-and-down volatility is often called noise.)

Q 2.55 If the interest rate is 5% per annum, how long will it take to double your money? How long will it take to triple it?

Q 2.56 If the interest rate is 8% per annum, how long will it take to double your money?

Q 2.57 From Fibonacci's *Liber Abaci*, written in the year 1202: "A certain man gave 1 denaro at interest so that in 5 years he must receive double the denari, and in another 5, he must have double 2 of the denari and thus forever. How many denari from this 1 denaro must he have in 100 years?"

Q 2.58 A bank quotes you a loan interest rate of 14% on your credit card. If you charge $15,000 at the beginning of the year, how much will you have to repay at the end of the year?

Q 2.59 Go to the website of a bank of your choice. What kind of quote does your bank post for a CD, and what kind of quote does your bank post for a mortgage? Why?

Q 2.60 What is the 1-year discount factor if the interest rate is 33.33%?

Q 2.61 You can choose between the following rent payments:

(a) A lump sum cash payment of $100,000;
(b) 10 annual payments of $12,000 each, the first occurring immediately;
(c) 120 monthly payments of $1,200 each, the first occurring immediately. (Friendly suggestion: This is a lot easier to calculate on a computer spreadsheet.)
(d) Which rental payment scheme would you choose if the interest rate was an effective 5% per year?
(e) Spreadsheet question: At what interest rate would you be indifferent between the first and the second choice above? (Hint: Graph the NPV of the second project as a function of the interest rate.)

Q 2.62 A project has cash flows of $15,000, $10,000, and $5,000 in 1, 2, and 3 years, respectively. If the prevailing interest rate is 15%, would you buy the project if it costs $25,000?

Q 2.63 Consider the same project that costs $25,000 with cash flows of $15,000, $10,000, and $5,000. At what prevailing interest rate would this project be profitable? Try different interest rates, and plot the NPV on the y-axis, and the interest rate on the x-axis.

Q 2.64 On April 12, 2006, Microsoft stock traded for $27.11 and claimed to pay an annual dividend of $0.36. Assume that the first dividend will be paid in 1 year, and that it then grows by 5% each year for the next 5 years. Further, assume that the prevailing interest rate is 6% per year. At what price would you have to sell Microsoft stock in 5 years in order to break even?

Q 2.65 Assume you are 25 years old. The IAW insurance company is offering you the following retirement contract (called an *annuity*): Contribute $2,000 per year for the next 40 years.

When you reach 65 years of age, you will receive \$30,000 per year for as long as you live. Assume that you believe that the chance that you will die is 10% per year after you will have reached 65 years of age. In other words, you will receive the first payment with probability 90%, the second payment with probability 81%, and so on. Assume the prevailing interest rate is 5% per year, all payments occur at year-end, and it is January 1 now. Is this annuity a good deal? (Use a spreadsheet.)

Q 2.66 A project has the following cash flows in periods 1 through 4: −\$200, +\$200, −\$200, +\$200. If the prevailing interest rate is 3%, would you accept this project if you were offered an upfront payment of \$10 to do so?

Q 2.67 Assume you are a real estate broker with an exclusive contract—the condo association rules state that everyone selling their condominiums must go through you or a broker designated by you. A typical condo costs \$500,000 today and sells again every 5 years. This will last for 50 years, and then all bets are off. Your commission will be 3%. Condos appreciate in value at a rate of 2% per year. The interest rate is 10% per annum.

(a) What is the value of this exclusivity rule? In other words, at what price should you be willing to sell the privilege of exclusive condo representation to another broker?

(b) If free Internet advertising was equally effective and if it could replace all real estate brokers so that buyers' and sellers' agents would no longer earn the traditional 6% (3% each), what would happen to the value gain of the condo?

Q 2.68 If the interest rate is 5% per annum, what would be the equivalent annual cost (see Question 2.39) of a \$2,000 lease payment up front, followed by \$800 for three more years?

Q 2.69 The prevailing discount rate is 15% per annum. Firm *F*'s cash flows start with \$500 in year 1 and grow at 20% per annum for 3 years. Firm *S*'s cash flows also start with \$500 in year 1 but shrink at 20% per annum for 3 years. What are the prices of these two firms? Which one is the better "buy"?

Investor Choice: Risk and Reward

We are still after the same prize: a good estimate of the corporate cost of capital ($\mathcal{E}(\tilde{r})$) in the NPV formula. But before you can understand the opportunity costs of capital for your firm's own projects, you have to understand the other opportunities that your investors have. This means that you must understand better what investors like (reward) and what they dislike (risk), how they are likely to measure their risks and rewards, how diversification works, what portfolios smart investors are likely to hold, and why "market beta" is a good measure of the contribution of an investment asset to the market portfolio's risk.

8.1 MEASURING RISK AND REWARD

We work with five assets that have four equally likely outcomes.

Put yourself into the shoes of an investor and start with the most basic questions: How should you measure the risk and reward of your portfolio? As always, we first cook up a simple example and then generalize our insights into a broader real-world context. Let's follow four risky assets (securities), named A through D, plus a risk-free asset named F. These assets could even be portfolios, themselves consisting of many individual portfolios, assets, and so on. (This is essentially what a mutual fund is.)

Historical samples can be viewed as scenarios.

There are four equally likely scenarios, named S1 through S4, as in Table 8.1. (If you find it easier to think in terms of historical outcomes, you can pretend that scenario S1 happened at time 1, S2 at time 2, and so forth, and you are now analyzing this historical data. This is not entirely correct, but often a helpful metaphor.) Which investment strategies do you deem better or worse, safer or riskier? It is the goal of this section to analyze the assets and scenarios in Table 8.1 to sharpen your understanding of the concepts and trade-offs of risk and reward.

➤ Why this is not entirely correct, Section 8.5A, p. 223

In a histogram, bars to the right mean higher returns. Bars that are more spread out indicate higher risk.

Visuals always help, so Figure 8.1 graphs the returns in Table 8.1. Each scenario is equally likely (the histogram bars are all equally tall), so you can just indicate where each outcome lies on the x-axis. In this histogram plot, you prefer assets that have scenario outcomes farther to the right (they have higher rates of return), outcomes

TABLE 8.1 RATES OF RETURN ON FIVE INVESTMENT ASSETS

	Assets' Rates of Return r				
Future	Pfio A (M)	Pfio B	Pfio C	Pfio D	Pfio F
In Scenario S1 ♣	−1.0%	+2.0%	−2.0%	+14.0%	+1.0%
In Scenario S2 ♦	+2.0%	+11.0%	+3.0%	+6.0%	+1.0%
In Scenario S3 ♥	+4.0%	−1.0%	+7.0%	0.0%	+1.0%
In Scenario S4 ♠	+11.0%	+4.0%	+12.0%	−12.0%	+1.0%
"Reward" ($\mathcal{E}(\tilde{r})$)	4.0%	4.0%	5.0%	2.0%	1.0%
"Variance" $\mathcal{V}ar(\tilde{r})$	19.5%%	19.5%%	26.5%%	90.0%%	0.0%%
"Risk" ($\mathcal{S}dv(\tilde{r})$)	4.42%	4.42%	5.15%	9.49%	0.00%

We use *pfio* as an abbreviation for *portfolio*. Variance ($\mathcal{V}ar$) and standard deviation ($\mathcal{S}dv$) were explained in Section 6.1B. The four scenarios are "stand-ins" for a much larger and exhaustive set of possible outcomes that could occur. For illustration, we assume that they are the only possible outcomes and that it is perfectly known that they occur with equal probability.

➤ Random variables *are* histograms, Section 6.1A, p. 138

that are *on average* farther to the right (they have higher *expected* rates of return), and outcomes that are more bunched together (they have less risk). Visual inspection shows that investment F has outcomes perfectly bunched at the same spot, so it is not only least risky but also risk-free. It is followed by the risky A and B, then C, and finally, the most risky, D.

8.1A MEASURING REWARD: THE EXPECTED RATE OF RETURN

Measure reward with the expected rate of return.

Although graphical measures are helpful, we also need algebraic formulas with associated numerical measures. A good measure for the **reward** is easy: You can use the **expected rate of return**, which is the probability-weighted average of all possible returns. For example, the mean rate of return for asset A is

$$\mathcal{E}(\tilde{r}_A) = (1/4)\cdot(-1\%) + (1/4)\cdot(+2\%) + (1/4)\cdot(+4\%) + (1/4)\cdot(+11\%)$$
$$= +4\%$$
$$\mathcal{E}(\tilde{r}_A) = \mathcal{P}rob(S1)\cdot(\tilde{r}_{S1}) + \mathcal{P}rob(S2)\cdot(\tilde{r}_{S2}) + \mathcal{P}rob(S3)\cdot(\tilde{r}_{S3}) + \mathcal{P}rob(S4)\cdot(\tilde{r}_{S4})$$

If you invest in A, you would expect to earn a rate of return of 4%. Because each outcome is equally likely, you can compute this faster as a simple average:

$$\mathcal{E}(\tilde{r}_A) = \frac{(-1\%) + (+2\%) + (+4\%) + (+11\%)}{4} = 4\%$$

8.1B MEASURING RISK: THE STANDARD DEVIATION OF THE RATE OF RETURN

Measure risk with the standard deviation of the rate of return.

➤ Measures of risk (standard deviation), Section 6.1B, p. 141

A good measure of risk is less obvious than a good measure of reward. You already got a good peek at the most common risk measures in Section 6.1B, but let's do it again in the context of our specific set of securities. Figure 8.1 shows that A is more spread out than F (i.e., A is more risky than F) and less spread out than D (i.e., A is less risky than

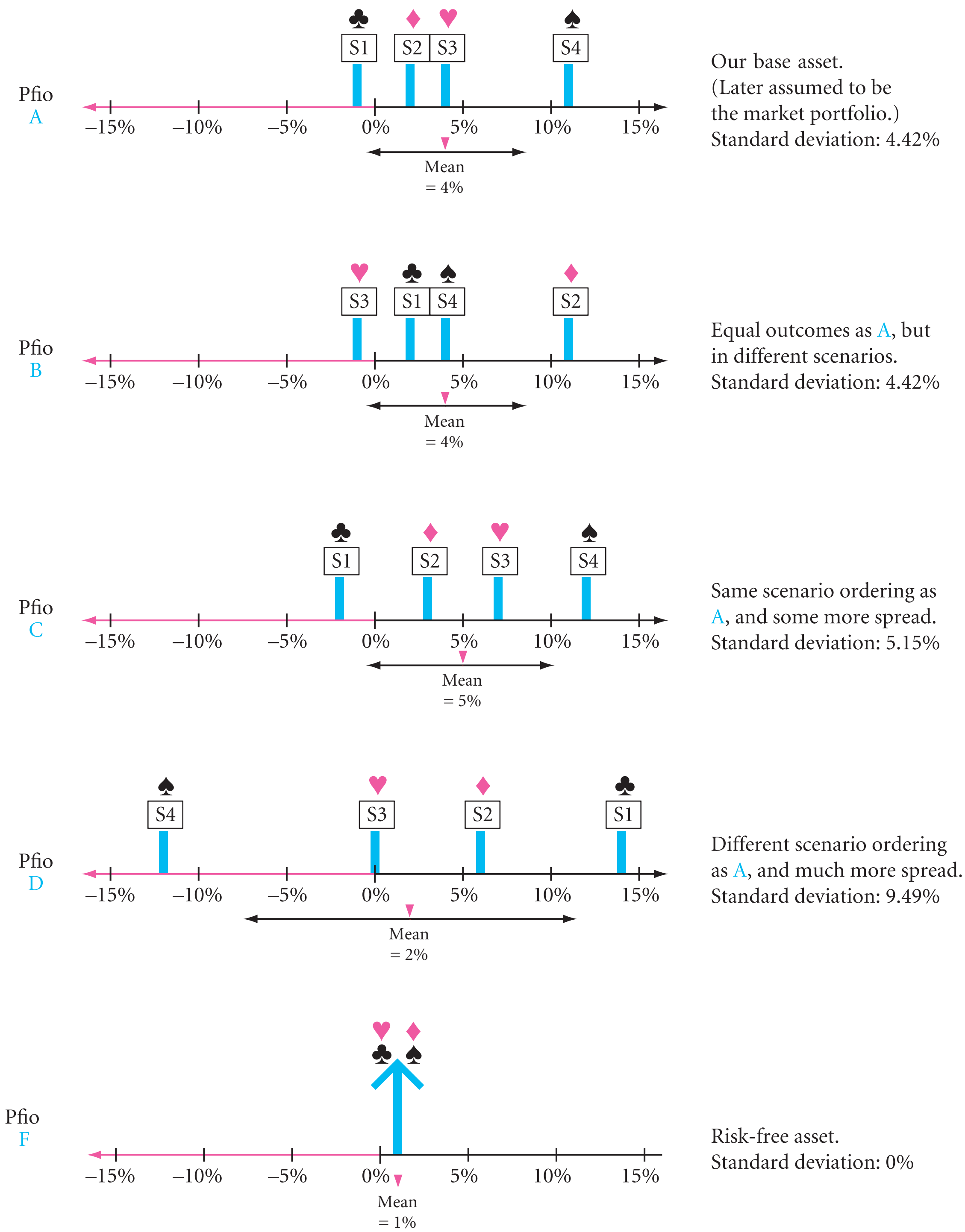


FIGURE 8.1 Graphical Perspectives on Performance

The graphs are standard histograms. Each outcome is equally likely, so each bar is 25% tall—with the exception of the bar in the final graph for the risk-free security, which is 100% tall. ♣ is the rate of return outcome in scenario S1, the ♦ in scenario S2, the ♥ in scenario S3, and the ♠ in scenario S4. The left-right arrows below the axes indicate the standard deviations.

D). A good first intuition is that it would make sense to rate each data point by how far away it is from the center (average). If your average is +4%, an outcome of 3% would be closer to the mean than an outcome of 0%. The former is only 1 unit away from the mean. The latter is 4 units away from the mean. It therefore makes sense to think in such deviations from the mean. Here is how A shapes up in terms of deviations from its mean:

Outcomes	In S1 (♣)	In S2 (♦)	In S3 (♥)	In S4 (♠)
Asset A Rate of Return	−1%	+2%	+4%	+11%
. . . in deviation from its 4% mean	−5%	−2%	0%	+7%

The average deviation from the mean is always 0. It cannot measure risk.

Unfortunately, you cannot compute risk as the average deviation from the mean. It is always zero—for example, the average here is $(-5 - 2 + 0 + 7)/4 = 0$. You must "neutralize" the sign, so that negative deviations count the same as positive deviations. The common fix is to compute the average *squared* deviation from the mean. This is called the **variance**:

$$\mathcal{V}ar(\tilde{r}_{\text{A}}) = \frac{(-1\% - 4\%)^2 + (2\% - 4\%)^2 + (4\% - 4\%)^2 + (11\% - 4\%)^2}{4}$$

$$= \frac{(-5\%)^2 + (-2\%)^2 + (0\%)^2 + (+7\%)^2}{4} = 19.5\%\%$$

$$= \frac{[r_{S1} - \mathcal{E}(\tilde{r})]^2 + [r_{S2} - \mathcal{E}(\tilde{r})]^2 + [r_{S3} - \mathcal{E}(\tilde{r})]^2 + [r_{S4} - \mathcal{E}(\tilde{r})]^2}{\text{Number of Outcomes}}$$

$$= \frac{\text{Sum over All Scenarios } \left[\tilde{r}_S \text{ in Scenario } S - \mathcal{E}(\tilde{r})\right]^2}{\text{Number of Scenarios}}$$

The variance has units that are intrinsically impossible to interpret for ordinary humans (% *squared* $= 0.01 \cdot 0.01$, written as $x\%\%$). Therefore, the variance carries very little intuition, except that a higher variance means more risk.

The standard deviation of the portfolio's rate of return is a common measure of risk.

A measure that has more meaningful units is the **standard deviation**. It is just the square root of the variance,

$$\mathcal{S}dv(\tilde{r}_{\text{A}}) = \sqrt{\mathcal{V}ar(\tilde{r}_{\text{A}})} = \sqrt{19.5\%\%} \approx 4.42\% \qquad (8.1)$$

*The standard deviation of the portfolio's rate of return is the most common measure of overall **portfolio risk**.* Looking at Figure 8.1, you can see that this standard deviation of 4.42% seems like a reasonable measure of how far the typical outcome of A is away from the mean of A. The last row in Table 8.1 also lists the standard deviations of B–F. In Figure 8.1, you can see their visual representations: F is risk-free; A and B are equally risky at 4.42%; C is a little more risky at 5.15%; and D is most risky at 9.49%.

IMPORTANT:

- You can measure investment portfolio reward by the expected rate of return on the *overall* portfolio.
- You can measure investment portfolio risk by the standard deviation of the rate of return on the *overall* portfolio.

(Warning: You will not measure the investment risk *contributions* of individual assets inside the portfolio via their standard deviations. This will be explained in Section 8.3B.)

At this point, you should begin to wonder how risk and reward are related in a reasonable world. This will be the subject of much of the next chapter. The brief answer for now is that you can speculate in dumb ways that give you high investment risk with low reward—as anyone who has gambled knows. However, if you are smart, after eliminating all investment mistakes (the low-hanging fruit), you have no choice but to take on more risk if you want to earn higher rewards.

A preview: Smart investors eliminate unnecessary risk. After they have done so, more reward requires taking more risk.

SOLVE NOW!

Q 8.1 What happens if you compute the average deviation from the mean, rather than the average squared deviation from the mean?

Q 8.2 Asset A from Table 8.1 offers −1%, +2%, +4%, and +11% with equal probabilities. Now add 5% to each of these returns. This new asset offers +4%, +7%, +9%, and +16%. Compute the expected rate of return, the variance, and the standard deviation of this new asset. How does it compare to A?

Q 8.3 Compute the risk and reward of C from Table 8.1.

8.2 PORTFOLIOS, DIVERSIFICATION, AND INVESTOR PREFERENCES

In the real world, you are usually not constrained to purchase assets in isolation—you can purchase a little of each. This has the important consequence of reducing your overall portfolio risk. Let's see why.

Start again with investment assets A and B, which offer the same rates of return, but in different future scenarios. If you purchase \$100 in either A or B, you would expect to earn \$4 with a risk of \$4.42. But what if you purchase \$50 in A and \$50 in B? Call this your investment portfolio P. In this case, your \$100 investment would look like this:

Portfolios are bundles of multiple assets. Their returns can be averaged.

Scenario Outcome	S1 (♣)	S2 (♦)	S3 (♥)	S4 (♠)	Average
Return on \$50 in A:	\$49.50	\$51.00	\$52.00	\$55.50	\$52.00
Return on \$50 in B:	\$51.00	\$55.50	\$49.50	\$52.00	\$52.00
⇒ Total return in P:	\$100.50	\$106.50	\$101.50	\$107.50	\$104.00
Cost = \$100 ⇒ Rate of return in P:	0.5%	6.5%	1.5%	7.5%	4.0%

You can do this more quickly by using the returns on A and B themselves. In this case, your portfolio P invests portfolio weight $w_A = 50\%$ into A and $w_B = 50\%$ in B. For example, to obtain the 6.5% in scenario S2, you could have computed the portfolio rate of return from A's 2% rate of return and B's 11% rate as

$$\tilde{r}_P = \tilde{r}_{50\% \text{ in A}, 50\% \text{ in B (in S2)}} = 50\% \cdot 2\% \quad + \quad 50\% \cdot 11\% = 6.5\%$$

$$\tilde{r}_{P=(w_1, w_2, \ldots, w_N)} \quad = \quad w_1 \cdot \tilde{r}_1 \quad + \cdots + \quad w_N \cdot \tilde{r}_N$$

Thus, you could have computed P's four scenario rates of return as follows:

$$\text{In S1 ♣:} \quad r_{P=(50\% \text{ in A}, 50\% \text{ in B}) \text{ in S1}} = 50\% \cdot (-1\%) + 50\% \cdot (+2.0\%) = 0.5\%$$

$$\text{In S2 ♦:} \quad r_{P=(50\% \text{ in A}, 50\% \text{ in B}) \text{ in S2}} = 50\% \cdot (+2\%) + 50\% \cdot (+11.0\%) = 6.5\%$$

$$\text{In S3 ♥:} \quad r_{P=(50\% \text{ in A}, 50\% \text{ in B}) \text{ in S3}} = 50\% \cdot (+4\%) + 50\% \cdot (-1.0\%) = 1.5\%$$

$$\text{In S4 ♠:} \quad r_{P=(50\% \text{ in A}, 50\% \text{ in B}) \text{ in S4}} = 50\% \cdot (+11\%) + 50\% \cdot (+4.0\%) = 7.5\%$$

$$r_{\text{P in S}} \quad = \quad w_A \cdot r_{\text{A in S}} \quad + \quad w_B \cdot r_{\text{B in S}}$$

Visually, the A and B combination portfolio called P has lower variability (risk and range) than either A or B.

Now look at these three possible investment portfolios: A, B, and P. The four outcomes are plotted sequentially on the x-axis in Figure 8.2, as if they had occurred in different months. (As noted, this is not a bad way to think about scenarios.) Each box has the name of its portfolio in it. The gray and magenta areas from −1% to +11% are the outcome ranges for single investments in either A or B. However, the investment of half in A and half in B has a much smaller range of outcomes (from 0.5% to 7.5%), as shown by the blue area. This portfolio P simply has less variability and range than either of its two components.

Algebraically, the combination portfolio also has lower risk.

Can you use the algebraic measures to back up your visual perception? The average (expected) rate of return of the combination portfolio P is the same 4% as that of A and B. The risk of the combination portfolio P is lower, however, than the risk of either A or B. In fact, it is

$$\mathcal{V}ar_{\substack{50\% \text{ in A} \\ 50\% \text{ in B}}} = \frac{(0.5\% - 4\%)^2 + (6.5\% - 4\%)^2 + (1.5\% - 4\%)^2 + (7.5 - 4\%)^2}{4}$$

$$= 9.25\%\%$$

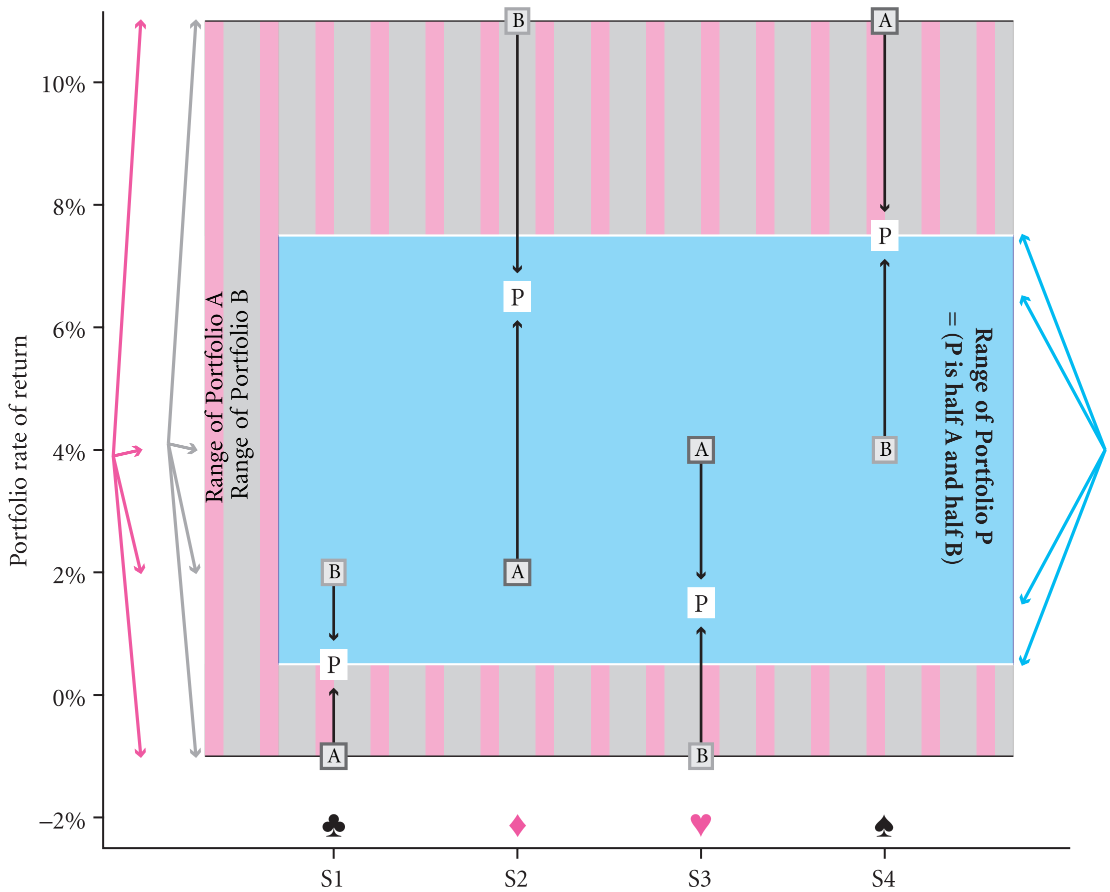
$$= \frac{[r_{S1} - \mathcal{E}(\tilde{r})]^2 + [r_{S2} - \mathcal{E}(\tilde{r})]^2 + [r_{S3} - \mathcal{E}(\tilde{r})]^2 + [r_{S4} - \mathcal{E}(\tilde{r})]^2}{N}$$

$$\Longrightarrow \quad \mathcal{S}dv_{50\% \text{ in A}, \ 50\% \text{ in B}} = \sqrt{\mathcal{V}ar} = \sqrt{9.25\%\%} \approx 3.04\%$$

An investment in either A or B has a risk of 4.42%. But an investment in half of A and half of B has a risk of only 3.04%! Why?

This is caused by diversification.

The reason is **diversification**—the mixing of different investments within a portfolio that reduces the impact of each one on the overall portfolio performance. More simply put, diversification means that not all of your eggs are in the same basket. If one investment component goes down, the other investment component sometimes



	Portfolio Rates of Return, r		
Future	Pfio A	Pfio B	Pfio P = AB
In Scenario S1 ♣	−1.0%	+2.0%	+0.5%
In Scenario S2 ♦	+2.0%	+11.0%	+6.5%
In Scenario S3 ♥	+4.0%	−1.0%	+1.5%
In Scenario S4 ♠	+11.0%	+4.0%	+7.5%
“Reward” ($\mathcal{E}(\tilde{r})$)	4.0%	4.0%	4.0%
“Risk” ($\mathcal{S}dv(\tilde{r})$)	4.42%	4.42%	**3.04%**
“Range” (see figure)	12%	12%	7%

Portfolio P is half A, half B. Because each half-A/half-B point is halfway between A and B, P has lower spread (risk) than either of its components, A and B, by itself. (The risks of A and B were computed as 4.42% in Formula 8.1 on page 204.) Returns on the single-asset portfolios A and B range from −1% to +11%, i.e., 12%. Returns on the combination-asset portfolio P range from +0.5% to +7.5%, i.e., 7%. This is color-coded as blue in the figure. The “spread arrows” on the left and the right also point to the possible outcomes of the three portfolios, indicating the variability visually. The combination portfolio has less spread.

FIGURE 8.2 Rate of Return Outcomes for A, B, and the 50%-50% Combination Portfolio P happens to go up, or vice versa. The imperfect correlation ("nonsynchronicity") reduces the overall portfolio risk.

8.2A ASSUME INVESTORS CARE ONLY ABOUT RISK AND REWARD

Investors love diversification: the more the better. They could like the market portfolio because it is highly diversified.

This intuition suggests that heavily diversified portfolios—portfolios that invest in many different assets—tend to have lower risk. As a corporate manager, it would be reasonable to assume that your investors are smart. Because diversification helps them reduce their investment risk, you can also reasonably believe that they are indeed holding heavily diversified portfolios. The most heavily diversified portfolio contains a little of every asset. Therefore, we often assume that your investors' portfolio is the overall **market portfolio**, consisting of all available investment opportunities.

If your investors like high reward and low risk and hold the market portfolio, you can work out how your projects affect them.

Why would you want to make any assumptions about your investors' portfolios? The answer is that if you are willing to assume that your investors are holding the market (or something very similar to it), your job as a corporate manager becomes much easier. Instead of asking what each and every one of your investors might possibly like, you can just ask, "When would my investors want to give me their money for investment into my firm's project, given that my investors are currently already holding the broad overall stock market portfolio?" The answer will be as follows:

1. Your investors should like projects that offer more reward (higher expected rates of return).
2. Your investors should like projects that help them diversify away some of the risk in the market portfolio, so that their overall portfolios end up being less risky.

In sum, your corporate managerial task is to take those projects that your investors would like to add to their current (market) portfolios. You should therefore search for projects that have high expected rates of return and high diversification benefits with respect to the market. Let's now turn toward measuring this second characteristic: How can your projects aid your investors' diversification, and how should you measure how good this diversification is?

IMPORTANT:

- Diversification is based on imperfect correlation, or "nonsynchronicity," among investments. It helps smart investors reduce the overall portfolio risk.
- Therefore, as a corporate manager, absent other intelligence, you should believe that your investors tend to hold diversified portfolios. They could even hold portfolios as heavily diversified as the "entire market portfolio."
- As a corporate manager, your task is to think about how a little of your project can aid your investors in terms of its contribution to the risk and reward of their heavily diversified overall portfolios. (You should not think about how risky your project is in itself.)

SOLVE NOW!

Q 8.4 A combination portfolio named AB invests 90% in A and 10% in B.
(a) Compute its risk and reward.
(b) In a bar plot similar to those in Figure 8.1, would this new AB portfolio look less spread out than the P(50%, 50%) portfolio that was worked out in the table in Figure 8.2?

8.3 HOW TO MEASURE RISK CONTRIBUTION

If we are willing to assume that our smart investors are holding all assets in the market, then what projects offer them the best diversification?

8.3A AN ASSET'S OWN RISK IS NOT A GOOD MEASURE FOR ITS RISK CONTRIBUTION TO A PORTFOLIO

Comovement determines risk contribution.

Obviously, diversification does *not* help if two investment opportunities always move in the same direction. For example, if you try to diversify one $50 investment in A with another $50 investment in A (which always has the same outcomes), then your risk does not decrease. On the other hand, if two investment opportunities always move in *opposite* directions, then diversification works extremely well: One is a buffer for the other.

Pretend A is the market, now called M. Is C or D a better addition?

Let's formalize this intuition. For explanation's sake, assume that the stock market portfolio held by your investors is A from Table 8.1, so rename it M (for "market"). Assume that C and D are two projects that your firm could invest in, but you cannot choose both. C offers not only a higher expected rate of return than D (5% versus 2%) but also lower risk (5.15% versus 9.49%). As a manager, acting on behalf of your investors, would you therefore assume that project C is automatically better for them than D?

The combination MC has almost the same risk as M.

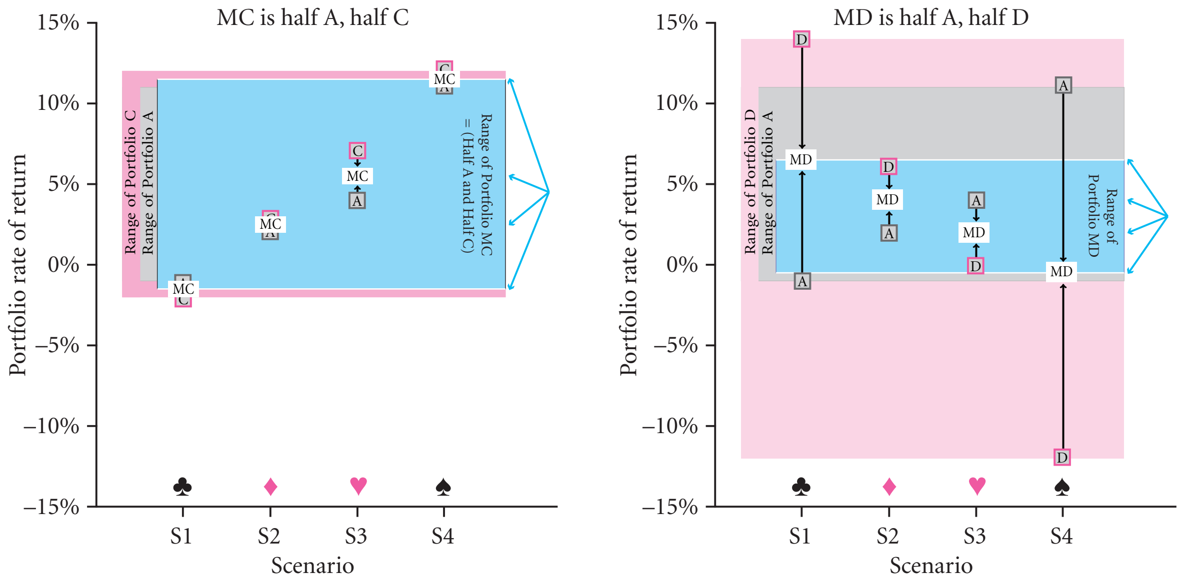
The answer is no. Let's assume that your investors start out with the market portfolio. Figure 8.3 shows what happens if they sell half of their M portfolio to invest in either C or D. You can call these two 50-50 portfolios MC and MD, respectively. Start with MC. If your investors reallocate half their money from M into C, their portfolios would have the following rates of return:

Scenario Outcome	$1 (♣)	S2 (♦)	S3 (♥)	S4 (♠)	Reward	Risk
MC	−1.5%	+2.5%	+5.5%	+11.5%	4.5%	4.74%

The left graph in Figure 8.3 plots the MC rates of return, plus the rates of return for both M and C by themselves. The magenta area is the range of portfolio C, the gray area is the range of portfolio A (i.e., M), and the blue area is the range of the combination portfolio. There is not much change in the risk of your portfolio in moving from a pure M portfolio to the MC portfolio. The risk actually increases from 4.42% to 4.74%.

The combination MD has much lower risk than M.

Now consider the combination of MD, which is the right graph in Figure 8.3. The pink area shows that, by itself, D is a very risky investment. However, if your investors



Portfolio Rates of Return r

Probability	Future	Pfio M (=A)	Pfio C	Pfio D	Pfio MC	Pfio MD
1/4	In Scenario S1 ♣	−1.0%	−2.0%	+14.0%	−1.5%	+6.5%
1/4	In Scenario S2 ♦	+2.0%	+3.0%	+6.0%	+2.5%	+4.0%
1/4	In Scenario S3 ♥	+4.0%	+7.0%	0.0%	+5.5%	+2.0%
1/4	In Scenario S4 ♠	+11.0%	+12.0%	−12.0%	+11.5%	−0.5%
	"Reward" ($\mathcal{E}(\tilde{r})$)	+4.00%	+5.00%	+2.00%	+4.50%	+3.00%
	"Risk" ($\mathcal{S}dv(\tilde{r})$)	4.42%	5.15%	9.49%	4.74%	2.57%
	Range	12%	14%	26%	13%	7%

Although the single-asset portfolio D is much riskier than the single-asset portfolio C, D is much better than C in reducing the risk of the market portfolio. This is because D tends to move opposite to the market. This can also be seen by looking at the color-coded ranges. The magenta range of C in the left figure is smaller than the pink range of D in the right figure. C is simply less risky an investment *in itself* than D. However, the blue range in the MC portfolio on the left is much bigger than the blue range in the MD portfolio on the right. As a portfolio component combined with the market portfolio, D adds much less risk than C. The arrows on the right of each figure also point to the possible outcomes of the combination portfolios, and help indicate their spreads.

FIGURE 8.3 Combining M with either C or D

instead reallocate half of their wealth from M into D, their overall portfolio would have the following rates of return:

Scenario Outcome	S1 (♣)	S2 (♦)	S3 (♥)	S4 (♠)	Reward	Risk
MD	+6.5%	+4.0%	+2.0%	−0.5%	3.0%	2.57%

This is much lower than the range of outcomes on the left (with portfolio C). The MD combination portfolio is simply much safer—even though D by itself is much riskier. If you compare the MC spread with the MD spread, the latter is much smaller. The algebraic risk measure, the standard deviation, confirms this: Even though D by itself is the riskiest choice, adding it to the M portfolio has reduced your investors' risk from 4.42% to 2.57%. In sum:

Portfolio	Reward	Risk	Note
M (=A) alone	4.00%	4.42%	The market portfolio that your investors were holding.
C alone	5.00%	5.15%	C is less risky than D, if purchased by itself.
D alone	2.00%	9.49%	
MC: half M, half C	4.50%	4.74%	If C is added to M, portfolio risk barely goes down,
MD: half M, half D	3.00%	2.57%	**but if D is added to** M**, portfolio risk goes down a lot!**

You now know that D's own higher standard deviation (9.49%) compared to C's (5.15%) is not a good indication of whether D helps your investors reduce portfolio risk more or less than C. If your investors are primarily holding M, then a very risky project like D can allow them to build lower-risk portfolios. However, if your investors are not holding any assets other than D, they would not care about D's diversification benefits and only about D's own risk. Thus, as a manager, you cannot determine whether your investors would prefer you to invest in C or D unless you know their entire portfolios. (Moreover, it could also depend on how your investors would like you to trade off more overall reward against more overall risk.)

The implication for your project choices as a corporate manager: Everything else equal, D could better reduce portfolio risk for your investors despite its higher own risk.

IMPORTANT: A project's (own) standard deviation is not necessarily a good measure of how it effects the risk of your investors' portfolios. Indeed, it is possible that a project with a very high standard deviation by itself may actually help lower an investor's overall portfolio risk.

SOLVE NOW!

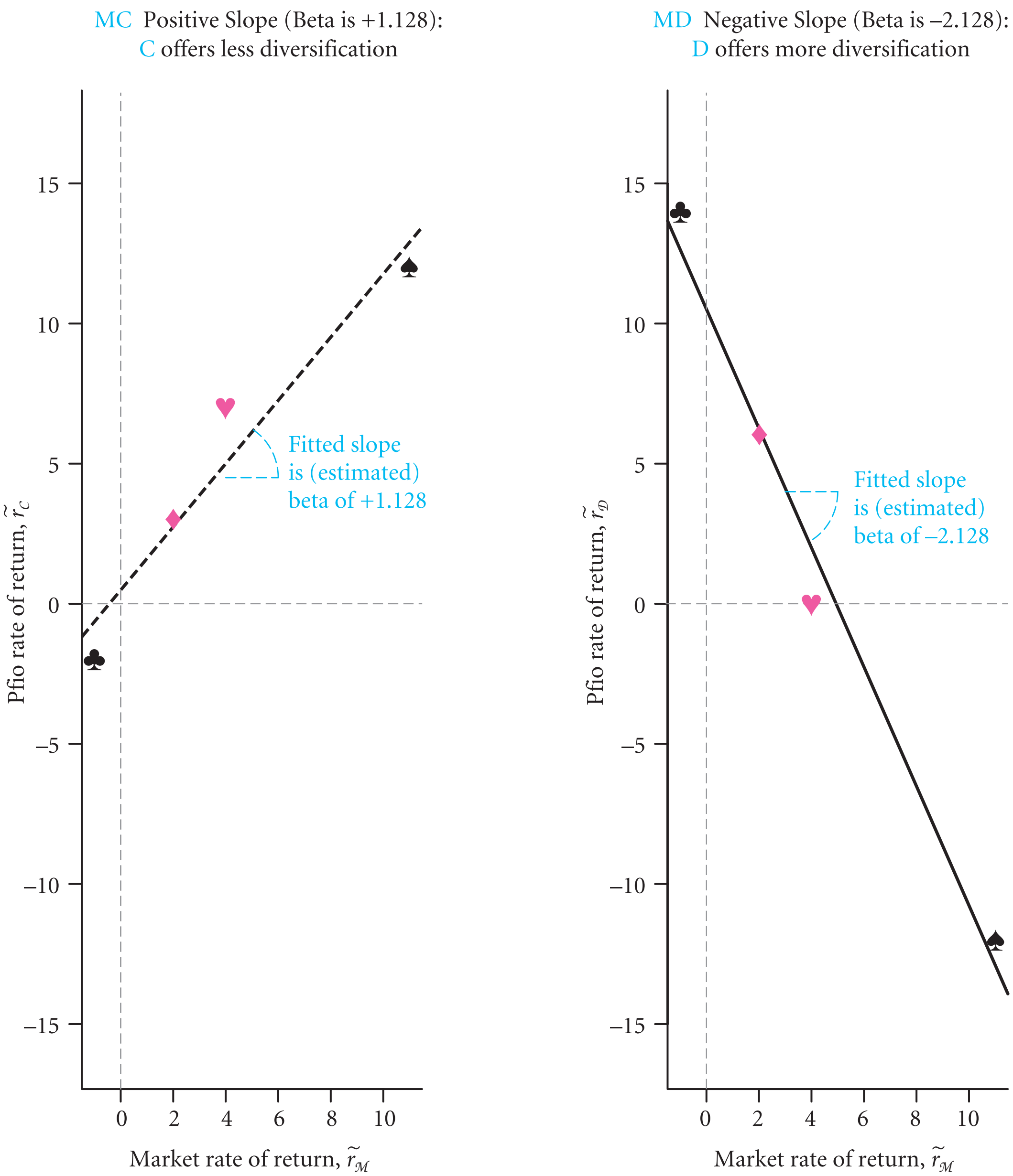
Q 8.5 Confirm the risk and reward calculations for the MC and MD portfolios in the table under Figure 8.3.

8.3B BETA IS A GOOD MEASURE FOR AN ASSET'S RISK CONTRIBUTION TO A PORTFOLIO

Can you guess why portfolio D is so much better than portfolio C in reducing the overall risk when held in combination with the M portfolio? The reason is that D tends to go up when M tends to go down, and vice versa. The same cannot be said for C—it tends to move together with M. You could call this "synchronicity" or "comovement." It is why C does not help investors who are heavily invested in the overall market in their quests to reduce their portfolio risks. Figure 8.4 shows the comovement graphically. If you draw the best-fitting line between M and C, the line slopes up. (It is the same kind of line that you already saw in Section 7.1C.) This means that C tends to be higher when M is higher. If you draw the best-fitting line between M and D, the line slopes down. This means that D tends to be higher when M is lower, and vice versa.

D reduces M's risk because it tends to move in the opposite direction. Comovement can be measured by the slope of a line.

➤ Market beta of PepsiCo, Section 7.1C, p. 185



The four data points in each plot are taken from Table 8.1 on page 202. They are the rates of return on the portfolios M, C, and D. These rates of return are quoted in percentages. In the example, you know that these are the four true possible outcomes. In the real world, if the four points were not the true known outcomes, but just the historical outcomes (sample points), then the slope would not be the true unknown beta, but only the "estimated" beta.

FIGURE 8.4 Possible Outcomes: Rates of Return versus Market Rate of Return

This slope is a common measure of expected comovement or countermovement—how much diversification benefit an investor can obtain from adding a particular new project. A higher slope means more comovement and less diversification; a lower, or even negative, slope means less comovement and more diversification.

The slope of a line is generally called a **beta** because it is common to write the formula for a line as

The line's slope is called beta.

$$y = \alpha + \beta \cdot x \tag{8.2}$$

A beta of 1 is a 45° diagonal line; a beta of 0 is a horizontal line. A positive beta slopes up; a negative beta slopes down; and a beta of infinity is a vertical line. The particular line we care about in finance is the **market beta**. It matters if you can posit that your smart investors are primarily holding the market portfolio. If so, you want to know how the rate of return on your own project comoves with that of the market. This is exactly what the market beta of your project tells you. To find it, draw the rate of return on M on the x-axis (hence the prefix "market" in market beta) and the rate of return on your project (here, either C or D) on the y-axis. Then take a ruler and try to draw the best line between the four points. You will find that the market beta of C is positive (your best line is upward sloping), whereas the market beta of D is negative (your best line is downward sloping). In statistics, you should have learned that you can find the beta by running a linear regression. If you don't remember, no worries: In Section 8.3B, I will show you again how to compute them exactly. For now, take my word that the two best lines are

We want the beta of the asset's rate of return on the y-axis and the market's rate of return on the x-axis.

$$\tilde{r}_C \approx 0.49\% + (+1.128) \cdot \tilde{r}_M \tag{8.3}$$

$$\tilde{r}_D \approx 10.51\% + (-2.128) \cdot \tilde{r}_M$$

$$\tilde{r}_i = \alpha_{i,M} + \beta_{i,M} \cdot \tilde{r}_M$$

The subscripts on the betas remind you what the variables on the x-axis and the y-axis are. The first subscript is on the y-axis, the second is on the x-axis, so $\beta_{C,M} \approx 1.128$ and $\beta_{D,M} \approx -2.128$. In fact, market beta plays such an important role in finance that the name "beta" has itself become synonymous for "market beta," and the second subscript is usually omitted. Formula 8.3 is sometimes called the **market model**.

IMPORTANT:

- Diversification works better if the new investment project tends to move in the opposite direction of the rest of the portfolio than if it tends to move in the same direction.
- It is often reasonable to assume that smart investors are already holding the market portfolio and are now considering investing into just a little of one additional asset—your firm's new project.
- If this new investment asset has a negative beta with respect to the market (its "market beta"), it means that it tends to go down when the market goes up, and vice versa. If this new investment asset has a positive beta with respect to the market, it means that it tends to move together with the market. If this new investment asset has a zero beta with respect to the market, it means that it moves independently of the market for all practical purposes.

- The market beta is a good measure of an investment asset's risk contribution for an investor who holds the market portfolio. The lower (or negative) the market beta, the more this investment helps reduce your investor's risk.
- The market beta of an asset can be interpreted as a line slope, where the rate of return on the market is on the *x*-axis and the rate of return on the new asset is on the *y*-axis. The line states how you expect the new asset to perform as a function of how the market will perform.
- You can think of market beta as a measure for "toxicity." In a reasonable equilibrium, holding everything else constant, risk-averse investors who are holding the market portfolio would agree to pay more for assets that have lower market betas. They would pay less for assets with higher market betas.

Warning: All of this beta-related risk measuring is interesting only if your investors are holding (portfolios close to) the overall market.

Before we conclude, some caveats are in order. From your perspective as the manager of a company, perhaps a publicly traded one, it is reasonable to assume that your investors are holding the market portfolio. It is also reasonable to assume that your new project is just a tiny new additional component of your investors' overall portfolios. We will staunchly maintain these assumptions, but you should be aware that they may not always be appropriate. If your investors are *not* holding something close to the market portfolio, then your project's market beta would *not* be a good measure of your projects' risk contributions. In the extreme, if your investors are holding *only* your project, market beta would not measure the project's risk contribution at all. This is often the case for entrepreneurs. They often have no choice but to put all their money into one egg in one basket. Such investors care only about the project's standard deviation, not the project's market beta.

Alpha has meaning, too, even though you won't use it just yet.

Although we shall not use it further in this book, the alpha intercept in Formula 8.3 also plays an important role. Together, alpha and beta help determine how attractive an investment is. For example, if the rate of return on the market is 10%, Formula 8.3 tells you that you would expect the rate of return on D to be

$$\mathcal{E}\left[(\tilde{r}_{\mathrm{D}}) | \text{ if } \tilde{r}_{\mathrm{M}} = 10\%\right] \approx 10.51\% + (-2.128) \cdot 10\% \approx -10.77\%$$

The higher the alpha, the better the average performance of your investment given any particular rate of return on the market. Just as investment professionals often call the market beta just beta, they often call this specific intercept (here 10.51%) just alpha. (There is one small complication: They usually subtract the risk-free interest rate first from both $\tilde{r}_{\mathrm{D}}$ and $\tilde{r}_{\mathrm{M}}$ in their regressions.)

Computing Market Betas from Historical Rates of Return

You can compute the best-fit beta via a 4-step procedure.

➤ Table 8.1, p. 202

Now that you understand what beta means, how can you actually compute it? Let me show you. Let's return to the assets in Table 8.1. What is the market beta of asset C? I have already told you that this slope is +1.128. To calculate it, I followed a tedious, but not mysterious, recipe. Here is what you have to do:

1. Just as you did for your variance calculations, first translate all returns into deviations from the mean. That is, for M, C, and D, subtract their own means from every realization.

First, de-mean each rate of return.
➤ Variance calculations, Section 6.1B, p. 141

	Original Rates of Return			Net-of-Mean Rates of Return		
Future	Pfio M	Pfio C	Pfio D	Pfio M	Pfio C	Pfio D
In Scenario S1 ♣	−1.0%	−2.0%	+14.0%	−5.0%	−7.0%	+12.0%
In Scenario S2 ♦	+2.0%	+3.0%	+6.0%	−2.0%	−2.0%	+4.0%
In Scenario S3 ♥	+4.0%	+7.0%	0.0%	0.0%	+2.0%	−2.0%
In Scenario S4 ♠	+11.0%	+12.0%	−12.0%	+7.0%	+7.0%	−14.0%
"Reward" ($\mathcal{E}(\tilde{r})$)	4.00%	5.00%	2.00%	0.00%	0.00%	0.00%

(How demeaning!)

2. Compute the variance of the series on the x-axis. This is the variance of the rates of return on M. With the net-of-mean M returns, this is easy:

Take squares and then average. This is the variance.

$$\mathcal{V}ar(\tilde{r}_{\text{M}}) = \frac{(-5\%)^2 + (-2\%)^2 + (0)^2 + (7\%)^2}{4} = 19.5\%\%$$

$$= \frac{\text{Sum over All Scenarios } S\text{: } [\tilde{r}_{\text{M}} \text{ in Scenario } S - \text{Average } \mathcal{E}(\tilde{r}_{\text{M}})]^2}{N}$$

Because 1% is "multiply by 0.01," 19.5%% could be rewritten as 0.195% or 0.00195. (Note also that you do not need to compute the variances of either C or D to obtain their market betas.)

3. Compute the average product of the net-of-mean variables. In this case, you want to compute the market beta for C, so you work with the rates of return on M and C.

For covariances, multiply net-of-mean returns, then average.

$$\mathcal{C}ov(\tilde{r}_{\text{M}}, \tilde{r}_{\text{C}}) = \frac{(-5\%) \cdot (-7\%) + (-2\%) \cdot (-2\%) + (0) \cdot (2\%) + (7\%) \cdot (7\%)}{4}$$

$$= 22\%\% = 0.22\% \quad (8.4)$$

$$= \frac{\text{Sum over All Scenarios } S\text{: } [\tilde{r}_{\text{M}} \text{ in Scenario } S - \mathcal{E}(\tilde{r}_{\text{M}})] \cdot [\tilde{r}_{\text{C}} \text{ in Scenario } S - \mathcal{E}(\tilde{r}_{\text{C}})]}{N}$$

This statistic is called the **covariance** between the rates of return on M and C.

4. The beta of C with respect to the market M, formally $\beta_{\text{C,M}}$ but often abbreviated as β_{C}, is the ratio of these two quantities,

The beta is the covariance divided by the variance.

$$\beta_{\text{C}} = \beta_{\text{C,M}} = \frac{22\%\%}{19.5\%\%} \approx 1.128 \quad (8.5)$$

$$= \frac{\mathcal{C}ov(\tilde{r}_{\text{M}}, \tilde{r}_{\text{C}})}{\mathcal{V}ar(\tilde{r}_{\text{M}})}$$

You can confirm our calculations using a spreadsheet.

This slope of 1.128 (a little more than a perfect 45° diagonal) is exactly the market beta we drew in Figure 8.4. Many spreadsheets and all statistical programs can compute it for you: They call the routine that does this a **linear regression**.

Think of market beta as the characteristic of an asset.

You should always think of the beta of an asset i with respect to a portfolio P as a characteristic measure of your asset i relative to an underlying base portfolio P. The rate of return on P is on the x-axis; the rate of return on i is on the y-axis. As we stated earlier, most often—but not always—the portfolio P is the market portfolio, M, so $\beta_{i,M}$ is often just called the market beta, or even just the beta (and the second subscript is omitted).

The average beta of the market (all stocks) is 1, not 0.

Now think for a moment. What should the average beta of a stock in the economy be? Equivalently, what is the beta of the market portfolio itself? Replace the C in Formula 8.5 with M:

$$\beta_M = \frac{\mathcal{Cov}(\tilde{r}_M, \tilde{r}_M)}{\mathcal{Var}(\tilde{r}_M)}$$

If you look at the definition of covariance, you can see that the covariance of a variable with itself *is* the variance. (The covariance is a generalization of the variance concept from one variable to two variables.) Therefore, $\mathcal{Cov}(\tilde{r}_M, \tilde{r}_M) = \mathcal{Var}(\tilde{r}_M)$, and the market beta of the market itself is 1. Graphically, if both the x-axis and the y-axis are graphing the same values, every point must lie on the diagonal. Economically, this should not be surprising, either: the market goes up one-to-one with the market.

Why torture you with computations? So you can play with scenarios.

Now that you know how to compute betas and covariances, you can consider scenarios for your project. For example, you might have a new project for which you would guess that it will have a rate of return of −5% if the market returns −10%; a rate of return of +5% if the market returns +5%; and a rate of return of 30% if the market returns 10%. Knowing how to compute a market beta therefore makes it useful to think of such scenarios. (You can also use this technique to explore the relationship between your projects and some other factors. For example, you could determine how your projects covary with the price of oil to learn about your project's oil risk exposure.)

➤ An oil-price beta, Section 9.8A, p. 292

Practical advice to help you estimate market beta in the real world: Use 3–5 years of daily observations and then adjust.

In the real world, you will sometimes think in terms of such scenarios, but more often you have to compute a market beta from historical rates of return for the overall stock market and for your project (or similar projects). Fortunately, as we noted up front, the beta computations themselves are exactly the same. In effect, when you use historical data, you simply assume that each time period was one representative scenario and proceed from there. Nevertheless, there are some real-world complications you should think about:

1. Should you use daily, weekly, monthly, or annual rates of return? The answer is that the best market beta estimates come from daily or weekly data. Annual data should be avoided (except in a textbook in which space is limited). Monthly data can be used if need be.
2. How much data should you use? Most researchers tend to use 3–5 years of historical rate of return data. This reflects a trade-off between having enough data and not going too far back into ancient history, which may be less relevant. If you have daily data, 3 years works quite well.

3. Is the historical beta a good estimate of the future beta? It turns out that history can sometimes be deceptive, especially if your estimated historical beta is far away from the market's beta average of 1. Fortunately, there are at least two methods to help adjust historical betas so that you get better estimates of future betas:
   (A) **Averaging:** You could rely not just on the historical beta computed from your own project's returns. Instead, you could use the average historical betas for many other projects that are similar to your own (for example, projects from the same industry or in the same size class). Such averages are usually less noisy.
   (B) **Shrinking:** You could "shrink" your historical beta toward the overall market beta of 1. For example, in the simplest such shrinker, you would simply compute an average of the overall market beta of 1 and your historical market beta estimate. If you computed a historical market beta of 4 for your project, you would work with a prediction of future market beta of about $(4 + 1)/2 = 2.5$ for your project.

   Many smart executives start with a statistical beta estimated from historical data and then use their intuitive judgment to adjust it.

SOLVE NOW!

Q 8.6 Return to your computation of market beta of 1.128 in Formula 8.5. We called it $\beta_{C,M}$, or β_C for short. Is the order of the subscripts important? That is, please compute $\beta_{M,C}$ and see whether it is also 1.128.

8.3C WHY NOT CORRELATION OR COVARIANCE?

Covariance and beta (and correlation) always have the same sign.

There is a close family relationship between covariance, beta, and correlation. The beta is the covariance divided by one of the variances. The correlation is the covariance divided by both standard deviations. The denominators are always positive. Thus, if the covariance is positive, so are the beta and the correlation; if the covariance is negative, so are the beta and the correlation; and if the covariance is zero, so are the beta and the correlation. The nice thing about the correlation, which makes it useful in many contexts outside finance, is that it has no scale and is always between −100% and +100%:

- Two variables that always move perfectly in the same direction have a correlation of 100%.
- Two variables that always move perfectly in opposite directions have a correlation of −100%.
- Two variables that are independent have a correlation of 0%.

This makes correlations very easy to interpret. The not-so-nice thing about correlation is that it has no scale and is always between −100% and +100%. This means that two investments, the second being a million times bigger than the first (all project rates of return multiplied by a million), have the same correlation with the stock market. Yet, the second investment would go up or down with any slight tremor in the market by a million times more, which would of course mean that it would contribute

TABLE 8.2 SOME MARKET BETAS AND CAPITALIZATIONS ON MAY 10, 2008

		Mkt-	Market Beta				Mkt-	Market Beta	
Company	Ticker	Cap [a]	Yahoo	AOL	Company	Ticker	Cap [a]	Yahoo	AOL
AMD	AMD	4	1.96	2.67	Intel	INTC	124	1.73	1.85
Coca-Cola	KO	130	0.52	0.78	PepsiCo	PEP	107	0.24	0.22
Citigroup	C	124	1.71	1.31	J.P. Morgan	JPM	158	0.85	0.91
Goldman Sachs	GS	74	2.24	1.84	Morgan Stanley	MS	51	1.75	1.71
IBM	IBM	170	0.95	0.94	Hewlett-Packard	HPQ	121	1.09	1.46
Dell	DELL	39	1.53	1.36	Sun	JAVA	10	1.01	2.13
Apple Inc	AAPL	162	2.86	2.57	Sony	SNE	45	0.97	1.08
Google	GOOG	180	2.60	2.17	Yahoo	YHOO	36	0.39	1.03
Ford	F	16	2.11	2.13	General Motors	GM	12	1.50	1.64
American Airlines	AMR	2	1.71	2.69	Southwest	LUV	9.5	−0.12	0.13
Exxon Mobil	XOM	469	1.14	1.04	Barrick Gold	ABX	34	−0.20	0.49
Philip Morris	PM	109	0.00	NA	Procter & Gamble	PG	199	0.63	0.57
Textron	TXT	15	1.46	1.81	Boeing	BA	63	1.08	1.22

a. "Mkt-Cap" is the equity market value in billions of dollars. Yahoo explained its betas as follows:

> The Beta is beta of equity. Beta is the monthly price change of a particular company relative to the monthly price change of the S&P 500. The time period for Beta is 5 years when available, and not less than 2.5 years. This value is updated monthly.

Note that Yahoo! *Finance* seems to ignore dividends, but this usually makes little difference. I could not find an explanation for the market betas provided by AOL. Google's market betas were the same as AOL's, but there was no explanation for them, either.

much more risk. The correlation ignores this, which disqualifies it as a serious candidate for a project risk measure. Fortunately, beta takes care of scale—indeed, the beta for the second project would be a million times larger. This is why we prefer beta over correlation as a measure of risk contribution to a portfolio.

8.3D INTERPRETING TYPICAL STOCK MARKET BETAS

Market beta works well when investors are holding the market and adding only a little of your project.

The market beta is the best measure of "diversification help" for an investor who holds the stock market portfolio and considers adding *just a little* of your firm's project. From your perspective as a manager seeking to attract investors, this is not a perfect, necessarily true assumption—but it is a reasonable one. Recall that we assume that investors are smart, so presumably they are holding highly diversified portfolios. To convince your market investors to like your $10 million project, you just need the average investor to want to buy $10 million divided by about $20 trillion (the stock market capitalization), which is 1/2,000,000 of their portfolios. For your investors, your corporate projects are just tiny additions to their market portfolios.

Most financial websites publish market beta estimates.

You can look up the market betas of publicly traded stocks on many financial websites. Table 8.2 lists the betas of some randomly chosen companies in May 2008 from Yahoo! *Finance* and from AOL's finance site. Most company betas are in the range of around 0 to about 2.5. A beta above 1 is considered risk-increasing for an investor holding the overall stock market (it is riskier than the stock market itself), while a beta below 1 is considered risk-reducing. Betas that are negative are quite rare—in Table 8.2, there happen to be only two such stocks, Southwest and Barrick Gold, according to the Yahoo betas, and none according to the AOL betas.

Beta can be viewed as the marginal change of your project with respect to the market.

Market beta has yet another nice intuitive interpretation: It is the degree to which the firm's value tends to change if the stock market changes. For example, Apple's market beta of approximately 2.7 (somewhere between 2.57 and 2.86) says that if the stock market will return an extra 5% next year (above and beyond its expectations), Apple will return an extra $2.7 \cdot 5\% = 13.5\%$ (above and beyond Apple's expectations). Of course, market beta is not a measure of how good an investment Apple is. (This measure is the alpha [which can be interpreted as an expected rate of return]. In the next section, you will learn a model that relates market beta to the expected rate of return. For now let's assume for illustration that there is no reasonable relationship.) Let's make the absurd assumption that Apple's expected rate of return is −30% and the more reasonable assumption that the market's expected rate of return is 10%. All that Apple's market beta then tells you is that when the market does 1% better than expected (i.e., $(10\% + 1\% = 11\%)$), then Apple would do 2.7% better than expected (i.e., $(-30\% + 2.7 \cdot 1\% = -27.3\%)$). If the market does 0% (i.e., 10% worse than expected), Apple would be expected to do −57% (i.e., 27% worse than expected). And so on. Apple's high market beta is useful because it informs you that if you hold the stock market, adding Apple stock would not help you diversify your market risk very much. Holding Apple stock would amplify any market swings, not reduce them. In any case, Apple's market beta does not tell you whether Apple is priced so high that it is an investment with a negative expected rate of return that you should avoid in the first place.

SOLVE NOW!

Q 8.7 You estimate your project x to return −5% if the stock market returns −10%, and +5% if the stock market returns +10%. What would you use as the market beta estimate for your project?

Q 8.8 You estimate your project to return +5% if the stock market returns −10%, and −5% if the stock market returns +10%. What would you use as the market beta estimate for your project?

8.4 EXPECTED RATES OF RETURN AND MARKET BETAS FOR (WEIGHTED) PORTFOLIOS AND FIRMS

Portfolios consist of multiple assets (or of other portfolios). Value-weighted and equal-weighted portfolios are defined.

Let's go back to your managerial perspective of figuring out the risk and return of your corporate projects. Many small projects are bundled together, so it is very common for managers to consider multiple projects already packaged together as one portfolio. For example, you can think of your firm as a collection of divisions that have been packaged together. If division C is worth \$1 million and division D is worth \$2 million, then a firm consisting of C and D is worth \$3 million. C constitutes 1/3 of the portfolio

"Firm" and D constitutes 2/3 of the portfolio "Firm." This kind of portfolio is called a **value-weighted portfolio** because the weights correspond to the market values of the components. (A portfolio that invests \$100 in C and \$200 in D would also be value-weighted. A portfolio that invests equal amounts in the constituents (for example, \$500 in each) is called an **equal-weighted portfolio**.)

What are the expected rate of return and market beta of a portfolio?

Thus, as a manager, you have to know how to work with a portfolio (firm) when you have all the information about all of its underlying component stocks (projects). If I tell you what the expected rate of return on each project is, and what the market beta of each project is, can you tell me what the firm's overall expected rate of return and overall market beta are? Let's try it. Use the C and D stocks from Table 8.1 on page 202, and call CDD the portfolio (or firm) that consists of 1/3 investment in division C and 2/3 investment in division D.

You can average *actual* rates of return.

Actually you already know that individual portfolio rates of return can be averaged. For example, in scenario S4 (♠), investment C has a rate of return of +12%, and investment D has a rate of return of −12%. Consequently, the overall investment CDD has a rate of return of

$$r_{\text{CDD, in S4 (♠)}} = 1/3 \cdot (+12\%) + 2/3 \cdot (-12\%) = -4\%$$
$$= w_C \cdot r_{\text{C, in S4}} + w_D \cdot r_{\text{D, in S4}}$$

Let us verify this: Put \$100 into C and \$200 into D. C turns into $1.12 \cdot \$100 = \$112$. D turns into $(1 - 12\%) \cdot \$200 = \$176$. The total portfolio turns into \$288, which is a rate of return of $\$288/\$300 - 1 = -4\%$ on a \$300 investment.

You can average *expected* rates of return.

It is also intuitive that *expected* rates of return can be averaged. In our example, C has an *expected* rate of return of 5%, and D has an *expected* rate of return of 2%. Consequently, your overall firm CDD has an expected rate of return of

$$\mathcal{E}(\tilde{r}_{\text{CDD}}) = 1/3 \cdot (+5\%) + 2/3 \cdot (+2\%) = 3\%$$
$$= w_C \cdot \mathcal{E}(\tilde{r}_C) + w_D \cdot \mathcal{E}(\tilde{r}_D)$$

Let us verify this, too. There are four possible outcomes: In S1, your actual rate of return is 8.67%; in S2, it is 5%; in S3, it is 2.33%; and in S4, it is −4%. The average of these four outcomes is indeed 3%.

News flash: You can also average market betas.

➤ Market betas of C and D, Formula 8.3, p. 213

But here is a remarkable and less intuitive fact: Market betas—that is, the projects' risk contributions to your investors' market portfolios—can be averaged, too. That is, I claim that the beta of CDD is the weighted average of the betas of C and D. You already computed the latter in Formula 8.3 as +1.128 and −2.128, respectively. Their value-weighted average is

$$\beta_{\text{CDD}} = 1/3 \cdot (+1.128) + 2/3 \cdot (-2.128) \approx -1.043 \tag{8.6}$$
$$= w_C \cdot \beta_C + w_D \cdot \beta_D$$

(*But* you cannot average variances or standard deviations!)

You will be asked in Q 8.9 to confirm this. However, do not think for a moment that you can compute value-weighted averages for all statistics. For example, variances and standard deviations cannot be averaged.

IMPORTANT:

- You can think of the firm as a weighted investment portfolio of components, such as individual divisions or projects. For example, if a firm named ab consists only of two divisions, a and b, then its rate of return is always

$$\tilde{r}_{ab} = w_a \cdot \tilde{r}_a + w_b \cdot \tilde{r}_b$$

where the weights are the relative values of the two divisions. (You can also think of this one firm as a "subportfolio" within a larger overall portfolio, such as the market portfolio.)

- The expected rate of return ("reward") of a portfolio is the weighted average expected rate of return of its components,

$$\mathcal{E}(\tilde{r}_{ab}) = w_a \cdot \mathcal{E}(\tilde{r}_a) + w_b \cdot \mathcal{E}(\tilde{r}_b)$$

Therefore, the expected rate of return of a firm is the weighted average rate of return of its divisions.

- Like expected rates of return, betas can be weighted and averaged. The beta of a firm—i.e., the firm's "risk contribution" to the overall market portfolio—is the weighted average of the betas of its components,

$$\beta_{ab} = w_a \cdot \beta_a + w_b \cdot \beta_b$$

Therefore, the market beta of a firm is the weighted average market beta of its divisions.

- You cannot do analogous weighted averaging with variances or standard deviations.

A firm is a portfolio of debt and equity. Thus, the portfolio formulas apply to the firm (with debt and equity as its components), too!

You can think of the firm not only as consisting of divisions, but also as consisting of debt and equity. For example, say your \$400 million firm is financed with debt worth \$100 million and equity worth \$300 million. If you own all debt and equity, you own the firm. What is the market beta of your firm's assets? Well, the beta of your overall firm must be the weighted average beta of its debt and equity. If your \$100 million in debt has a market beta of, say, 0.4 and your \$300 million of equity has a market beta of, say, 2.0, then your firm has a market beta of

$$1/4 \cdot (0.4) \quad + \quad 3/4 \cdot (2.0) \quad = 1.6$$

$$\beta_{\text{Firm}} = \left(\frac{\text{Debt value}}{\text{Firm value}}\right) \cdot \beta_{\text{Debt}} + \left(\frac{\text{Equity value}}{\text{Firm value}}\right) \cdot \beta_{\text{Equity}} \tag{8.7}$$

This 1.6 is called the **asset beta** to distinguish it from the **equity beta** of 2.0 that financial websites report. Put differently, if your firm refinances itself to 100% equity (i.e., \$400 million worth), then the reported market beta of your equity on Yahoo!

Finance would fall to 1.6. The asset beta is the measure of your firm's projects' risk contribution to the portfolio of your investors. It is the relevant measure that will determine the cost of capital that you should use as the hurdle rate for projects that are like the average project in your firm.

SOLVE NOW!

Q 8.9 Let's check that the beta combination formula (Formula 8.6 on page 220) is correct. Let me lead you along:

(a) Write down a table with the rate of return on the market and on portfolio CDD in each of the four possible states. (Hint: In scenario S1 [♣], the rate of return on CDD is 8.67%.) Then forget about C and D altogether. (In this question, you will work only with the market and CDD.)

(b) Compute the average rate of return on the market and on CDD.

(c) Write down a table with the de-meaned market rate of return and CDD rate of return in each of the four possible states. (The mean of the de-meaned returns must be zero.)

(d) Multiply the de-meaned rates of return in each scenario. This gives you four cross-products, each having units of %%. (Hint: In scenario S1 [♣], it is about −28.35%%.)

(e) Compute the average of these cross-products. This is the covariance between CDD and M.

(f) Divide the covariance between CDD and M by the variance of the market. Is it equal to the −1.04 from Formula 8.6?

(g) Which is faster—this route or Formula 8.6?

Q 8.10 Let's confirm that you cannot take a value-weighted average of component variances (and thus of standard deviations) the same way that you can take value-weighted average expected rates of return and value-weighted average market betas.

(a) What would the value-weighted average variance of CDD be?

(b) What is the actual variance of CDD?

Q 8.11 Consider an investment of 2/3 in C and 1/3 in D. Call this new portfolio CCD. Compute the variance, standard deviation, and market beta of CCD. Do this two ways: first from the four individual scenario rates of return of CCD, and then from the statistical properties of C and D itself.

Q 8.12 Assume that a firm will always have enough money to pay off its bonds, so the beta of its bonds is 0. (Being risk free, the rate of return on the bonds is obviously independent of the rate of return on the stock market.) Assume that the beta of the underlying assets is 2. What would financial websites report for the beta of the firm's equity if it changes its current capital structure from all equity to half debt and half equity? To 90% debt and 10% equity?

8.5 SPREADSHEET CALCULATIONS FOR RISK AND REWARD

Doing all these calculations by hand is tedious. We computed these statistics within the context of just four scenarios, so that you would understand their meanings better. However, you can do this faster in the real world. Usually, you would download reams of real historical rates of return data into a computer spreadsheet, like Excel or OpenOffice. Spreadsheets have all the functionality you need already built in—and you now understand what their functions actually calculate. In practice, you would use the following functions:

In real life, you can do calculations faster with a spreadsheet.

- **average**(*range*) computes the average (rate of return).
- **varp**(*range*) computes the (population) variance. If you worked with historical data instead of known scenarios, you would instead use the **var**(*range*) formula. (The latter divides by $N - 1$ rather than by N, which I will explain in a moment.)
- **stdevp**(*range*) computes the (population) standard deviation. If you used historical data instead of known scenarios, you would instead use the **stdev**(*range*) formula.
- **covar**(*range-1,range-2*) computes the population covariance between two series. If Excel was consistent, this function should be called covarp rather than covar.
- **correl**(*range-1,range-2*) computes the correlation between two series.
- **slope**(*range-Y,range-X*) computes a beta. If *range-Y* contains the rates of return of an investment and *range-X* contains the rates of return on the market, then this formula computes the market beta.

Table 8.3 shows a computer spreadsheet that computes everything that you did in this chapter.

8.5A STATISTICAL NUANCES

In this chapter, we have continued to presume (just as we did in Section 7.1E) that historical data gives us an unbiased guide to the future when it comes to means, variances, covariances, and betas. Of course, this is a simplification—and remember that it can be a problematic one. I already noted that this is less of a problem for covariances, variances, and betas than it is for means. Rely on historical means as predictors of future expected rates of return only at your own risk!

➤ Will history repeat itself?, Section 7.1E, p. 189

There is a second, smaller statistical issue that you should be aware of. Statisticians often use a covariance formula that divides by $N - 1$, not N. Strictly speaking, dividing by $N - 1$ is appropriate if you work with historical data. These are just sample draws and not the full population of possible outcomes. With a sample, you do not really know the true mean when you de-mean your observations. The division by a smaller number, $N - 1$, gives a larger but unbiased covariance estimate. It is also often called the *sample covariance*. In contrast, dividing by N is appropriate if you work with "scenarios" that you know to be true and equally likely. In this case, the statistic is often called the *population covariance*. The difference rarely matters in finance, where you usually have a lot of observations—except in our book examples where you have only four scenarios. (For example, dividing by $N = 1{,}000$ and by $N = 1{,}001$ gives almost the same number.)

When working with a *sample*, the (co)variance formula divides by $N-1$. When working with the *population*, the (co)variance formula divides by N.

TABLE 8.3 The Computer Spreadsheet

	A	B	C	D	E	F	G	H	I	J	K	L	M	Formula
1	Investor Choice, Sample Spreadsheet								Combinations with A (M)					
2									MB	MC	MD			
3		Base Portfolios							w_B	w_C	w_D		w_{CDD}	
4		A (M)	B	C	D		F		= 1/2	= 1/2	= 1/2		= (1/3, 2/3)	
5	S1	–1.0%	2.0%	–2.0%	14.0%		1.0%		= 0.5%	= –1.5%	= 6.5%		= 8.7%	
6	S2	2.0%	11.0%	3.0%	6.0%		1.0%		= 6.5%	= 2.5%	= 4.0%		= 5.0%	
7	S3	4.0%	–1.0%	7.0%	0.0%		1.0%		= 1.5%	= 5.5%	= 2.0%		= 2.3%	
8	S4	11.0%	4.0%	12.0%	–12.0%		1.0%		= 7.5%	= 11.5%	= –0.5%		= –4.0%	
9														
10														Formula
11	Average	= 4.0%	= 4.0%	= 5.0%	= 2.0%		= 1.0%		= 4.0%	= 4.5%	= 3.0%		= 3.0%	←=average(×5:×8)
12	Variance	= 0.1950%	= 0.1950%	= 0.2650%	= 0.9000%		= 0.0000%		= 0.093%	= 0.225%	= 0.066%		= 0.214%	←=varp(×5:×8)
13	Risk	= 4.42%	= 4.42%	= 5.15%	= 9.49%		= 0.000%		= 3.041%	= 4.743%	= 2.574%		= 4.625%	←=stdevp(×5:×8)
14														
15														
16	Market Beta	= 1.000	= –0.051	= 1.128	= –2.128		= 0.000		= 0.474	= 1.064	= –0.564		= –1.043	←=slope(×5:×8,B5:B8)
17														
18	Alpha	= 0.00%	= 4.21%	= 0.49%	= 10.51%		= 1.00%		= 2.10%	= 0.24%	= 5.26%		= 7.17%	←=intercept(×5:×8,B5:B8)
19	Correlation	= 100.0%	= –5.1%	= 96.8%	= –99.1%		= #DIV/0!		= 68.9%	= 99.1%	= –96.8%		= –99.6%	←=correl(×5:×8,B5:B8)
20	Covariance	= 0.20%	= –0.01%	= 0.22%	= –0.42%		= 0.00%		= 0.09%	= 0.21%	= –0.11%		= –0.20%	←=covar(×5:×8,B5:B8)

This spreadsheet (also available on the book website) demonstrates the main statistical calculations that are performed in this chapter. Please note that we are using the population variance and population standard deviation formulas, not the sample variance and sample standard deviation formulas. Spreadsheet cells that are formulas contain an '='. In rows 5–8, columns I–K are equal combinations of M (column B) and one other portfolio (B–D), which are in columns C–E, respectively. Column M is a weighted average of columns J and K. Formulas in rows 11–20 are given on the right side.

The only reason why you even needed to know this is that if you use a program that has a built-in variance or standard deviation function, you should not be surprised if you get numbers different from those you have computed in this chapter. In some programs, you can get both functions. In Excel, you can use the *varp* and *stdevp* population statistical functions to get the population statistics, not the *var* and *stdev* functions that would give you the sample statistics.

This is important to keep in mind if you use a spreadsheet to check your work.

Beta is not affected by whether you divide the variance/covariance by N or $N - 1$, because both numerator (covariance) and denominator (variance) are divided by the same number.

For market beta, the divisor cancels out and does not matter.

Furthermore, statisticians distinguish between underlying unknown statistics and statistics estimated from the data. For example, they might call the unknown true mean μ and the sample mean m (or $\bar{x}$). They might call the unknown true beta β^T and the estimated sample beta a beta with a little hat ($\hat{\beta}$). And so on. Our book is casual about the difference for lack of space, but keep in mind that whenever you work with historical data, you are really just working with sample estimates.

My fault: Our notation should have distinguished between true population and estimated sample statistics.

SUMMARY

This chapter covered the following major points:

- The expected rate of return is a measure of expected reward.

$$\mathcal{E}(\tilde{r}_P) = \frac{\text{Sum over All Scenarios: } \left[\text{Return of Pfio P in Each Scenario}\right]}{N}$$

- The variance is (roughly) the average squared deviation from the mean.

$$\mathcal{V}ar(\tilde{r}_P) = \frac{\text{Sum over All Scenarios: } \left[\text{Return of Pfio P in Each Scenario} - \mathcal{E}(\tilde{r}_P)\right]^2}{N - 1}$$

  Sometimes, you may divide by N instead of $N - 1$. (With a lot of data, this makes no difference.) The variance is an intermediate input to the more interesting statistic, the standard deviation.

- The standard deviation is the square root of the variance. The standard deviation of the rate of return of a portfolio is commonly used as the measure of its risk.

$$\mathcal{S}dv(\tilde{r}_P) = \sqrt{\mathcal{V}ar(\tilde{r}_P)}$$

- Diversification reduces the risk of a portfolio.
- We assume that investors are smart enough to hold widely diversified portfolios, which resemble the overall market portfolio. Diversified portfolios offer higher expected rates of return at lower risks compared to undiversified portfolios.
- An individual asset's own risk is not a good measure of its risk contribution to a portfolio.
- Market beta is a good measure of the risk contribution of an individual asset for an investor who holds the market portfolio.

- Market betas for typical stocks range between 0 and 2.5.
- It is a straightforward application of formulas to compute beta, correlation, and covariance. They are closely related and always share the same sign.
- Like expected rates of return, betas can be averaged (using proper weighting). However, variances or standard deviations cannot be averaged.

KEY TERMS

SOLVE NOW! SOLUTIONS

Q 8.1 The average deviation from the mean is always 0.

Q 8.2 The mean of portfolio A was 4%. Adding 5% to each return will give you a mean of 9%, which is 5% higher. The variance and standard deviation remain at the same level, the latter being 4.42%. If you think of 5% as a constant c, you have just shown that $\mathcal{E}(\tilde{r} + c) = \mathcal{E}(\tilde{r}) + c$ and $\mathcal{S}dv(\tilde{r} + c) = \mathcal{S}dv(\tilde{r})$.

Q 8.3 The reward of portfolio C is its expected rate of return. This is simply $[(-2\%) + 3\% + 7\% + 12\%]/4 = 5\%$. (We just divide by 4, rather than multiply each term by 1/4, because all outcomes are equally likely.) The variance of C is $[(-7\%)^2 + (-2\%)^2 + (2\%)^2 + (7\%)^2]/4 = 26.5\%\%$. The standard deviation, which is our measure of risk, is $\sqrt{26.5\%\%} \approx 5.15\%$.

Q 8.4 For the combination portfolio of 90% in A and 10% in B:

(a) The reward, that is, the expected rate of return, is $0.9 \cdot 4\% + 0.1 \cdot 4\% = 4\%$. To work out the variance, first compute the rates of return in the four states:

$$\begin{aligned} S1: &\ 0.9 \cdot (-1\%) + 0.1 \cdot (2\%) = -0.7\% \\ S2: &\ 0.9 \cdot (2\%) + 0.1 \cdot (11\%) = 2.9\% \\ S3: &\ 0.9 \cdot (4\%) + 0.1 \cdot (-1\%) = 3.5\% \\ S4: &\ 0.9 \cdot (11\%) + 0.1 \cdot (4\%) = 10.3\% \end{aligned}$$

The variance is

$$\frac{(-0.7\% - 4\%)^2 + (2.9\% - 4\%)^2 + (3.5\% - 4\%)^2 + (10.3\% - 4\%)^2}{4}$$

$$\approx \frac{22.09\%\% + 1.21\%\% + 0.25\%\% + +39.69\%\%}{4} = 15.81\%\%$$

The standard deviation is $\sqrt{15.81\%\%} \approx 3.98\%$.

(b) Figure 8.2 on page 207 showed that the risk (standard deviation) of the 50%-50% portfolio was 3.04%. The risk (standard deviation) of the 90%-10% portfolio is 3.98%. Thus, the latter portfolio looks more spread out in a bar plot.

Q 8.5 For the MC portfolio, the portfolio combination rates of return in the four scenarios were on the right side of the table in Figure 8.3 on page 210. Let's confirm them first:

$$\begin{aligned}
\text{In S1:} &\quad 0.5 \cdot (-1\%) + 0.5 \cdot (-2\%) = -1.5\% \\
\text{In S2:} &\quad 0.5 \cdot (2\%) + 0.5 \cdot (3\%) = 2.5\% \\
\text{In S3:} &\quad 0.5 \cdot (4\%) + 0.5 \cdot (7\%) = 5.5\% \\
\text{In S4:} &\quad 0.5 \cdot (11\%) + 0.5 \cdot (12\%) = 11.5\%
\end{aligned}$$

The expected rate of return is

$$\mathcal{E}(\tilde{r}_{\text{MC}}) = \frac{-1.5\% + 2.5\% + 5.5\% + 11.5\%}{4} = 4.5\%$$

The variance of this portfolio is

$$\mathcal{V}ar_{\text{MC}} = \frac{(-1.5\% - 4.5\%)^2 + (2.5\% - 4.5\%)^2 + (5.5\% - 4.5\%)^2 + (11.5\% - 4.5\%)^2}{4} = 22.5\%\%$$

Therefore, $\mathcal{S}dv_{\text{MC}} = \sqrt{22.5\%\%} \approx 4.74\%$.
For the MD portfolio,

$$\begin{aligned}
\text{In S1:} &\quad 0.5 \cdot (-1\%) + 0.5 \cdot (14\%) = 6.5\% \\
\text{In S2:} &\quad 0.5 \cdot (2\%) + 0.5 \cdot (6\%) = 4.0\% \\
\text{In S3:} &\quad 0.5 \cdot (4\%) + 0.5 \cdot (0\%) = 2.0\% \\
\text{In S4:} &\quad 0.5 \cdot (11\%) + 0.5 \cdot (-12\%) = -0.5\%
\end{aligned}$$

The expected rate of return is

$$\mathcal{E}(\tilde{r}_{\text{MD}}) = \frac{6.5\% + 4.0\% + 2.0\% - 0.5\%}{4} = 3\%$$

The variance is $\mathcal{V}ar_{\text{MD}} = [(6.5\% - 3\%)^2 + (4.0\% - 3\%)^2 + (2.0\% - 3\%)^2 + (-0.5\% - 3\%)^2]/4 =$ $6.625\%\%$. Therefore, $\mathcal{S}dv_{\text{MD}} = \sqrt{6.625\%\%} \approx 2.57\%$.

Q 8.6 The order of subscripts on market beta is important. Algebraically, $\beta_{\text{C,M}} = [\text{cov}(\tilde{r}_{\text{C}}, \tilde{r}_{\text{M}})]/[\text{var}(\tilde{r}_{\text{M}})]$, while $\beta_{\text{MC}} = [\text{cov}(\tilde{r}_{\text{C}}, \tilde{r}_{\text{M}})]/[\text{var}(\tilde{r}_{\text{C}})]$. The denominator is different. The easiest way to compute the latter is to pick off the standard deviation of 5.15% from Table 8.1 and square it ($26.52\%\% = 0.2652\%$). Therefore, the beta is

$$\beta_{\text{M,C}} = \frac{\mathcal{C}ov(\tilde{r}_{\text{M}}, \tilde{r}_{\text{C}})}{\mathcal{V}ar(\tilde{r}_{\text{C}})} \approx \frac{0.22\%}{0.2652\%} \approx 0.83$$

This is not the same as $\beta_{\text{C,M}} \approx 1.128$. Fortunately, you will never ever need to compute $\beta_{\text{M,C}}$. I only asked you to do this computation so that you realize that the subscript order is important.

Q 8.7 The market beta of this project is

$$\beta_{x,\text{M}} = \frac{\tilde{r}_{x,2} - \tilde{r}_{x,1}}{\tilde{r}_{\text{M},2} - \tilde{r}_{\text{M},1}} = \frac{(-5\%) - (+5\%)}{(-10\%) - (+10\%)} = +0.5$$

(This is not "half as volatile" because market beta is not a measure of volatility.)

Q 8.8 Using the same formula, the market beta is $[(+5\%) - (-5\%)]/[(-10\% - (+10\%)] = -0.5$.

Q 8.9 To check that Formula 8.6 on page 220 is correct, you must compute the market beta for CDD from the rates of return for the entire firm CDD.

(a) The second and third columns in the following table show the rates of return on the market and on CDD in each of the four states:

	Original Base Rates		Net-of-Mean Rates		
Scenario	$\tilde{r}_M$	$\tilde{r}_{CDD}$	$\tilde{r}_M$	$\tilde{r}_{CDD}$	Cross-product
In S1 (♣)	−1%	8.67%	−5%	5.67%	−28.35%%
In S2 (♦)	2%	5.00%	−2%	2.00%	−4.00%%
In S3 (♥)	4%	2.33%	0%	−0.67%	0.00%%
In S4 (♠)	11%	−4.00%	7%	−7.00%	−49.00%%
Mean	4%	3%	0%	0%	−20.33%%

(b) The average rates of return are in the last row of the table.
(c) The de-meaned rates of return are in the fourth and fifth columns.
(d) The cross-products are in the sixth column.
(e) The average cross-product is in the last row of the sixth column.
(f) Using Formula 8.6, the beta of investment CDD is

$$\beta_{CDD} \approx \frac{\mathcal{C}ov(\tilde{r}_M, \tilde{r}_C)}{\mathcal{V}ar(\tilde{r}_M)} = \frac{-0.2033\%}{0.195\%} \approx -1.04$$

(g) Formula 8.6 is a bit easier than this route. The advantage would be even more obvious if you had a few hundred securities and a few thousand trading days, and you already knew the market beta for each of them individually.

In any case, you have now confirmed that Formula 8.6 yielded the same result. You did not catch me in a lie.

Q 8.10 To confirm that you cannot value-weight variances (and thus standard deviations):

(a) The variance of $\tilde{r}_C$ was 26.5%%. The variance of $\tilde{r}_D$ was 90.0%%. The value-weighted average of one part variance of C and two parts variance of D is $w_C \cdot \tilde{r}_C + w_D \cdot \tilde{r}_D = 1/3 \cdot 26.5\%\% + 2/3 \cdot 90.0\%\% \approx 68.83\%$.
(b) The actual variance of CDD is $\mathcal{V}ar(\tilde{r}_{CDD}) \approx [(5.67\%)^2 + (2\%)^2 + (-0.67\%)^2 + (-7\%)^2]/4 \approx 85.6\%\%/4 \approx 21.4\%\%$.

Q 8.11 The CCD portfolio has rates of return of 3.3333%, 4.00%, 4.6667%, and 4.00% in the four states. De-meaned, this is −0.6667%, 0%, 0.6667%, and 0%. Therefore, the variance of CCD is $[(-0.6667\%)^2 + (0\%)^2 + (0.6667\%)^2 + (0\%)^2]/4 \approx 0.224\%\%$, and its standard deviation is 0.47%. The de-meaned rates of return on M are −5%, −2%, 0, and 7%. The cross-products of the de-meaned CCD rates of return with the de-meaned M rates of return are therefore 3.3333%%, 0, 0, and 0. Therefore, the covariance of CCD and M is $(3.3333\%\% + 0 \cdot 3)/4 \approx 0.8333\%\%$. The variance of the market is 19.5%%. Therefore, the market beta of CCD is $0.833/19.5 \approx 0.0427$. This was the first method. Now the second method: $\beta_{CCD} = w_C \cdot \beta_C + w_D \cdot \beta_D \approx 2/3 \cdot (+1.128) + 1/3 \cdot (-2.128) \approx 0.0427$.

Q 8.12 For a firm whose debt is risk free, the overall firm beta is $\beta_{Firm} = 0.5 \cdot \beta_{Equity} + 0.5 \cdot \beta_{Debt}$. Thus, $0.5 \cdot \beta_{Equity} + 0.5 \cdot 0 = 2$. Solve for $\beta_{Equity} = \beta_{Firm}/0.5 = 4$. For the (90%, 10%) case, the equity beta jumps to $\beta_{Equity} = 2/0.1 = 20$.

PROBLEMS

The [blue box] indicates problems available in myfinancelab

When not otherwise specified in these problems, questions refer to the named portfolios A through F from Table 8.1.

Q 8.13 Multiply each rate of return for A by 2.0. This portfolio offers −2%, +4%, +8%, and +22%. Compute the expected rate of return and standard deviation of this new portfolio. How do they compare to those of the original portfolio A?

Q 8.14 The following were the closing year-end prices of the Japanese stock market index, the Nikkei-225:

1984	11,474	1992	16,925	2000	13,786
1985	13,011	1993	17,417	2001	10,335
1986	18,821	1994	19,723	2002	8,579
1987	22,957	1995	19,868	2003	10,677
1988	29,698	1996	19,361	2004	11,489
1989	38,916	1997	15,259	2005	16,111
1990	24,120	1998	13,842	2006	17,225
1991	22,984	1999	18,934	2007	15,308

Assume that each historical rate of return was exactly one representative scenario (independent sample draw) that you can use to estimate the future. If a Japanese investor had purchased a mutual fund that imitated the Nikkei-225, what would her annual rates of return, compounded rate of return (from the end of 1984 to the end of 2007), average rate of return, and risk have been?

Q 8.15 Compute the value-weighted average of 1/3 of the standard deviation of C and 2/3 of the standard deviation of D. Is it the same as the standard deviation of a CDD portfolio of 1/3 C and 2/3 D, in which your investment rate of return would be $1/3 \cdot \tilde{r}_C + 2/3 \cdot \tilde{r}_D$?

Q 8.16 What are the risk and reward of a combination portfolio that invests 40% in A and 60% in B?

Q 8.17 Consider the following five assets, which have rates of return in six equally likely possible scenarios:

	Scenarios					
	Awful	Poor	Med.	Okay	Good	Great
Asset P1	–2%	0%	2%	4%	6%	10%
Asset P2	–1%	2%	2%	2%	3%	3%
Asset P3	–6%	2%	2%	3%	3%	1%
Asset P4	–4%	2%	2%	2%	2%	20%
Asset P5	10%	6%	4%	2%	0%	–2%

(a) Assume you can only purchase one of these assets. What are their risks and rewards?
(b) Supplement your previous risk-reward rankings of assets P1–P5 with those of combination portfolios that consist of half P1 and half of each of the other 4 portfolios, P2–P5. What are the risks and rewards of these four portfolios?
(c) Assume that P1 is the market. Plot the rates of return for P1 on the x-axis and the return for each of the other stocks on their own y-axes. Then draw lines that you think best fit the points. Do not try to compute the beta—just use the force (and your eyes), Luke. If you had to buy just a little bit of one of these P2–P5 assets, and you wanted to lower your risk, which would be best?

Q 8.18 Assume you have invested half of your wealth in a risk-free asset and half in a risky portfolio P. Is it theoretically possible to lower your portfolio risk if you move your risk-free asset holdings into another risky portfolio Q? In other words, can you ever reduce your risk more by buying a risky security than by buying a risk-free asset?

Q 8.19 Why is it so common to use historical financial data to estimate future market betas?

Q 8.20 Is it wise to rely on historical statistical distributions as our guide to the future?

Q 8.21 Look up the market betas of the companies in Table 8.2. Have they changed dramatically since May 2008, or have they remained reasonably stable?

Q 8.22 You estimate your project to return −20% if the stock market returns −10%, and +5% if the stock market returns +10%. What would you use as the market beta estimate for your project?

Q 8.23 Go to Yahoo! *Finance*. Obtain 2 years' worth of weekly rates of return for PepsiCo and for the S&P 500 index. Use a spreadsheet to compute PepsiCo's market beta.

Q 8.24 Consider the following assets:

	Scenario		
	Bad	Okay	Good
Market M	−5%	5%	15%
Asset X	−2%	−3%	25%
Asset Y	−4%	−6%	30%

(a) Compute the market betas for assets X and Y.
(b) Compute the correlations of assets X and Y with M.
(c) Assume you were holding only M. You now are selling off 10% of your M portfolio to replace it with 10% of either X or Y. Would an MX portfolio or an MY portfolio be riskier?
(d) Is the correlation indicative of which of these two portfolios ended up riskier? Is the market beta indicative?

Q 8.25 Compute the expected rates of return and the portfolio betas for many possible portfolio combinations (i.e., different weights) of C and D from Table 8.1 on page 202. (Your weight in D is 1 minus your weight in C.) Plot the two against one another. What does your plot look like?

Q 8.26 The following represents the probability distribution for the rates of return for next month:

Probability	Pfio P	Market M
1/6	−20%	−5%
2/6	−5%	+5%
2/6	+10%	0%
1/6	+50%	+10%

Compute by hand (and show your work) for all the following questions.
(a) What are the risks and rewards of P and M?
(b) What is the correlation of M and P?
(c) What is the market beta of P?
(d) If you were to hold 1/3 of your portfolio in the risk-free asset, and 2/3 in portfolio P, what would its market beta be?

Q 8.27 Download the historical prices for the S&P 500 index (~spx or ~gspc) and for VPACX (the *Vanguard Pacific Stock Index* mutual fund) from Yahoo! *Finance*, beginning January 1, 2004, and ending December 31 of last year. Load them into a spreadsheet and position them next to one another. Compute the historical rates of return. Compute the risk and reward. Compute VPACX's market beta with respect to the S&P 500 index. How do your estimates compare to the Fund Risk as noted by Yahoo! *Finance*?

Q 8.28 Download 5 years of historical monthly (dividend-adjusted) prices for Coca-Cola (KO) and the S&P 500 from Yahoo! *Finance*.
(a) Compute the monthly rates of return.
(b) Compute the average rate of return and risk of portfolios that combine KO and the S&P 500 in the following proportions: (0.0, 1.0), (0.2, 0.8), (0.4, 0.6), (0.6, 0.4), (0.8, 0.2), (1.0, 0.0). Then plot them against one another. What does the plot look like?
(c) Compute the market beta of Coca-Cola.

Q 8.29 Are historical covariances or means more trustworthy as estimators of the future?

Q 8.30 Why do some statistical packages estimate covariances differently (and different from those we computed in this chapter)? Does the same problem also apply to expected rates of return (means) and betas?

CHAPTER 8 APPENDIX

Trade-Off between Risk and Return

8.6 AN INVESTOR'S SPECIFIC TRADE-OFF BETWEEN RISK AND REWARD

This appendix develops the trade-off between risk and return. Although this is not central to the subject of corporate finance, it is central to the subject of investments. So, where are we and where are we going?

What is the optimal portfolio of assets? (A portfolio is a complete set of weights on all possible assets.)

- You already know that diversification reduces risk.
- Therefore, you know that you like diversification.
- You know that assets that covary negatively with the rest of your portfolio are particularly desirable from a diversification perspective.
- The beta of an asset with respect to a portfolio is its measure of "toxicity" in the context of the portfolio.

The question that you cannot yet answer is

- Exactly how much of each asset should you purchase?

For example, is it better to purchase 25% in A and 75% in B, or 50% in each? How do you determine good investment weights? What is your optimal investment portfolio?

Table 8.4 computes different combinations of two assets to get various portfolio risk-reward characteristics.

Let's make up two new base assets, H and I. (If you wish, you can think of these assets as themselves being portfolios containing many different stocks.) How do you find the best combination portfolio of H and I? Table 8.4 shows some of the portfolios you could put together. Let's confirm the numbers for at least one of these. Portfolio K invests $w_H = 1/3$ in H and $w_I = 2/3$ in I, which means it has the following possible outcomes:

$$\text{In Scenario S1 } \clubsuit \quad \tilde{r}_K = 1/3 \cdot (-6\%) + 2/3 \cdot (-12\%) = -10\%$$

$$\text{In Scenario S2 } \diamondsuit \quad \tilde{r}_K = 1/3 \cdot (+12\%) + 2/3 \cdot (+18\%) = +16\%$$

$$\text{In Scenario S3 } \heartsuit \quad \tilde{r}_K = 1/3 \cdot (0\%) + 2/3 \cdot (+24\%) = +16\%$$

$$\text{In Scenario S4 } \spadesuit \quad \tilde{r}_K = 1/3 \cdot (+18\%) + 2/3 \cdot (+6\%) = +10\%$$

$$\tilde{r}_K = w_H \cdot (\tilde{r}_H) + w_I \cdot (\tilde{r}_I)$$

The expected rate of return of this portfolio, given all possible future scenarios, is then

$$\mathcal{E}(\tilde{r}_K) = 1/4 \cdot (-10\%) + 1/4 \cdot (+16\%) + 1/4 \cdot (+16\%) + 1/4 \cdot (+10\%) = 8\%$$

$$\mathcal{E}(\tilde{r}) = \text{Sum over All Scenarios } S\text{: } \mathcal{P}rob(\text{Scenario } S) \cdot \text{Outcome in Scenario } S$$

To compute the variance of K, you follow the procedure laid out in Section 6.1B: First, take out the mean from the rates of return:

➤ Standard deviation, Section 6.1B, p. 141

In Scenario S1 ♣ $-10\% - 8\% = -18\%$

In Scenario S2 ♦ $+16\% - 8\% = +8\%$

In Scenario S3 ♥ $+16\% - 8\% = +8\%$

In Scenario S4 ♠ $+10\% - 8\% = +2\%$

$\tilde{r}_K - \mathcal{E}(\tilde{r}_K)$

Second, square them and compute the average:

$$\mathcal{V}ar(\tilde{r}_K) = \frac{(-18\%)^2 + (+8\%)^2 + (+8\%)^2 + (+2\%)^2}{4} = 114\%\% \qquad (8.8)$$

The risk is therefore $\mathcal{S}dv(\tilde{r}_J) = \sqrt{\mathcal{V}ar(\tilde{r}_K)} = \sqrt{114\%\%} \approx 10.68\%$. You have now confirmed the three statistics for portfolio K in Table 8.4: the 8% expected rate of return (reward), 114%% variance, and 10.68% standard deviation (risk).

Do you care about your portfolio's beta or your portfolio's standard deviation? Make sure you understand the answer to this question.

IMPORTANT:

- As an investor, you usually care only about your portfolio's standard deviation (risk). (You rarely ever care about the overall market beta of your asset holdings.)
- If you are the CFO of a firm that wants to get into the market portfolio, so that investors willingly buy your shares, then you do care about your single firm's market beta. You should not care primarily about your firm's own standard deviation (idiosyncratic risk), because your investors do not care about it. They can diversify away your firm's idiosyncratic risk.

TABLE 8.4 PORTFOLIOS USED TO ILLUSTRATE MEAN-VARIANCE COMBINATIONS

	Base Assets		Combination Portfolios				
	100% in Pfio H	100% in Pfio I	1/4 in H 3/4 in I	1/3 in H 2/3 in I	1/2 in H 1/2 in I	2/3 in H 1/3 in I	3/4 in H 1/4 in I
Pfio Name	H	I	J	K	L	M	N
In Scenario S1 ♣	−6.0%	−12.0%	−10.50%	−10.00%	−9.00%	−8.00%	−7.50%
In Scenario S2 ♦	+12.0%	+18.0%	+16.50%	+16.00%	+15.00%	+14.00%	+13.50%
In Scenario S3 ♥	0.0%	+24.0%	+18.00%	+16.00%	+12.00%	+8.00%	+6.00%
In Scenario S4 ♠	+18.0%	+6.0%	+9.00%	+10.00%	+12.00%	+14.00%	+15.00%
"Reward" ($\mathcal{E}(\tilde{r})$)	6.00%	9.00%	8.25%	8.00%	7.50%	7.00%	6.75%
"Variance" ($\mathcal{V}ar(\tilde{r})$)	90.0%%	189.0%%	128.8%%	114.0%%	92.2%%	81.0%%	79.3%%
"Risk" ($\mathcal{S}dv(\tilde{r})$)	9.49%	13.75%	11.35%	10.68%	9.60%	9.00%	8.91%

These are the two base assets (and their combinations) used to illustrate the mean-variance efficient frontier in Section 8.8.

SOLVE NOW!

Q 8.31 Confirm the portfolio variance and standard deviation if you invest in portfolio M ($w_H = 2/3$) in Table 8.4.

Q 8.32 Confirm the portfolio variance and standard deviation if you invest in portfolio N ($w_H = 3/4$) in Table 8.4.

8.7 A SHORTCUT FORMULA FOR THE RISK OF A PORTFOLIO

We want to write the portfolio variance as a function of the component investment weights. This is a common shortcut formula.

There is a shortcut formula that can make portfolio variance computations faster. This shortcut allows you to compute the variance of a portfolio as a function of the weights in each constituent asset. To use it, you need to know the covariances between all assets. The formula also avoids having to first work out the rate of return of the combination portfolio in each and every scenario—not a big deal when there are four scenarios, but a very big deal if you have a thousand daily observations, each of which can count as a scenario, and you want to consider many portfolios with various weights.

Example: We still need the covariance between H and I.

➤ Covariance computation, Section 8.3B, p. 214

For our two assets, you need only one extra number for the new variance shortcut formula: You have to compute the covariance between your two base portfolios, here H and I. You have already worked with the covariance in Section 8.3B. It is defined as the average product of the two net-of-mean returns. Subtract the mean (6% for H and 9% for I) from each scenario's realization:

	Portfolio H		Portfolio I	
In Scenario S1 ♣	$\tilde{r}_H - \mathcal{E}(\tilde{r}_H)$	$= -12\%$	$\tilde{r}_I - \mathcal{E}(\tilde{r}_I)$	$= -21\%$
In Scenario S2 ♦	$\tilde{r}_H - \mathcal{E}(\tilde{r}_H)$	$= +6\%$	$\tilde{r}_I - \mathcal{E}(\tilde{r}_I)$	$= +9\%$
In Scenario S3 ♥	$\tilde{r}_H - \mathcal{E}(\tilde{r}_H)$	$= -6\%$	$\tilde{r}_I - \mathcal{E}(\tilde{r}_I)$	$= +15\%$
In Scenario S4 ♠	$\tilde{r}_H - \mathcal{E}(\tilde{r}_H)$	$= +12\%$	$\tilde{r}_I - \mathcal{E}(\tilde{r}_I)$	$= -3\%$

Therefore,

$$\mathcal{C}ov(\tilde{r}_H, \tilde{r}_I) = \frac{(-12\%) \cdot (-21\%) + (+6\%) \cdot (+9\%) + (-6\%) \cdot (+15\%) + (+12\%) \cdot (-3\%)}{4} = +45\%\%$$

$$\mathcal{C}ov(\tilde{r}_H, \tilde{r}_I) = \frac{\text{Sum over All Scenarios (or Observations) } S\text{: } [\tilde{r}_{H,S} - \mathcal{E}(\tilde{r}_H)] \cdot [\tilde{r}_{I,S} - \mathcal{E}(\tilde{r}_I)]}{N} \tag{8.9}$$

H and I are positively correlated—these investments tend to move together. Intuitively, this means, for example, that if the rate of return on portfolio H exceeds its 6% mean, portfolio I will also tend to exceed its own 9% mean.

Without further ado, the box that follows gives the shortcut formula for two assets.

IMPORTANT: The variance of a portfolio P that consists only of A and B, that is, with returns of $\tilde{r}_P = w_A \cdot \tilde{r}_A + w_B \cdot \tilde{r}_B$, where w_A is the portfolio weight in component A, and w_B is the portfolio weight in component B, is

$$\mathcal{V}ar(\tilde{r}_P) = w_A^2 \cdot \mathcal{V}ar(\tilde{r}_A) + w_B^2 \cdot \mathcal{V}ar(\tilde{r}_B) + 2 \cdot w_A \cdot w_B \cdot \mathcal{C}ov(\tilde{r}_A, \tilde{r}_B) \quad (8.10)$$

Check whether this is correct. Try it out on portfolio K, which invests 1/3 in H and 2/3 in I:

$$\begin{aligned}\mathcal{V}ar(\tilde{r}_K) &= (1/3)^2 \cdot \mathcal{V}ar(\tilde{r}_H) + (2/3)^2 \cdot \mathcal{V}ar(\tilde{r}_I) + 2 \cdot (1/3) \cdot (2/3) \cdot \mathcal{C}ov(\tilde{r}_H, \tilde{r}_I) \\ &= (1/3)^2 \cdot 90\%\% + (2/3)^2 \cdot 189\%\% + 2 \cdot (1/3) \cdot (2/3) \cdot (+45\%\%) \\ &= 114\%\%\end{aligned}$$

This is the same result as we computed in Formula 8.8, so the shortcut indeed gives the correct answer.

The general formula comes with a good memorization aid.

One way to remember this formula—and the more general version with more than two securities—is to create a matrix of all your assets. It's simple. Write all your assets' names on both edges, their weights next to them, and write into each cell what is on the edges as well as a covariance between what's on the edges:

		A	B	C	...
		w_A	w_B	w_C	
A	w_A	$w_A \cdot w_A \cdot \mathcal{C}ov(\tilde{r}_A, \tilde{r}_A)$	$w_A \cdot w_B \cdot \mathcal{C}ov(\tilde{r}_A, \tilde{r}_B)$	$w_A \cdot w_C \cdot \mathcal{C}ov(\tilde{r}_A, \tilde{r}_C)$	
B	w_B	$w_B \cdot w_A \cdot \mathcal{C}ov(\tilde{r}_B, \tilde{r}_A)$	$w_B \cdot w_B \cdot \mathcal{C}ov(\tilde{r}_B, \tilde{r}_B)$	$w_B \cdot w_C \cdot \mathcal{C}ov(\tilde{r}_B, \tilde{r}_C)$	
C	w_C	$w_C \cdot w_A \cdot \mathcal{C}ov(\tilde{r}_C, \tilde{r}_A)$	$w_C \cdot w_B \cdot \mathcal{C}ov(\tilde{r}_C, \tilde{r}_B)$	$w_C \cdot w_C \cdot \mathcal{C}ov(\tilde{r}_C, \tilde{r}_C)$	
⋮					⋱

That's it. By the way, did you notice that if you have m securities, there are only m variance terms in this matrix (on the diagonal), but $m^2 - m$ covariance terms? For 500 assets, you have 500 variance cells and 249,500 covariance cells. Adding the next security to the portfolio would add 1 variance term and 500 covariance terms. It should suggest to you that it need not be far-fetched to believe that the covariance of assets—how they fit together—can be more important than their own variances.

Apply the formula to compute the variance of K again.

Now substitute our specific investment weights for portfolio K, which are $w_H = 1/3$, $w_I = 2/3$. Let me also show you that investments that you do not own (call a sample one J) just drop out of the formula:

		H	I	J	...
		1/3	2/3	0	
H	1/3	$1/3 \cdot 1/3 \cdot \mathcal{C}ov(\tilde{r}_H, \tilde{r}_H)$	$1/3 \cdot 2/3 \cdot \mathcal{C}ov(\tilde{r}_H, \tilde{r}_I)$	$1/3 \cdot 0 \cdot \mathcal{C}ov(\tilde{r}_H, \tilde{r}_J)$	
I	2/3	$2/3 \cdot 1/3 \cdot \mathcal{C}ov(\tilde{r}_I, \tilde{r}_H)$	$2/3 \cdot 2/3 \cdot \mathcal{C}ov(\tilde{r}_I, \tilde{r}_I)$	$2/3 \cdot 0 \cdot \mathcal{C}ov(\tilde{r}_I, \tilde{r}_J)$	
J	0	$0 \cdot 1/3 \cdot \mathcal{C}ov(\tilde{r}_J, \tilde{r}_H)$	$0 \cdot 2/3 \cdot \mathcal{C}ov(\tilde{r}_J, \tilde{r}_I)$	$2/3 \cdot 0 \cdot \mathcal{C}ov(\tilde{r}_J, \tilde{r}_J)$	
⋮					⋱

All cells with J just multiply everything with a zero, so they can be omitted. Next, use the fact that, by definition, the covariance of something with itself is its variance. So, the matrix is

		H	I
		1/3	2/3
H	1/3	$1/3 \cdot 1/3 \cdot 90\%\%$	$1/3 \cdot 2/3 \cdot 45\%\%$
I	2/3	$2/3 \cdot 1/3 \cdot 45\%\%$	$2/3 \cdot 2/3 \cdot 189\%\%$

Add up all the cells, and you have the variance of portfolio K.

$$\mathcal{V}ar(\tilde{r}_K) = 1/3 \cdot 1/3 \cdot 90\%\% + \ 1/3 \cdot 2/3 \cdot 45\%\%$$
$$2/3 \cdot 1/3 \cdot 45\%\% + 2/3 \cdot 2/3 \cdot 189\%\% = 114\%\%$$

Again, this is the correct answer that you already knew.

This formula is useful if you want to try thousands of different portfolios (investment weights).

For H and I, this formula is not any more convenient than computing the scenario or historical time series of portfolio returns first and then computing the variance of this one series. However, the formula is a lot more convenient if you have to compute the portfolio variance of thousands of different combinations of H and I and there are hundreds of scenarios. And it is precisely this process—recomputing the overall portfolio variance many times—that is at the heart of determining the best portfolio: You want to know how different portfolio weights change your portfolio risk. Your alternative to the shortcut would be to recompute the returns for each of the hundreds of possible portfolio weight combinations—which would quickly become very painful.

SOLVE NOW!

Q 8.33 Show that the shortcut Formula 8.10 works for portfolio M, in which H is 2/3. That is, does it give the same 81.0%% noted in Table 8.4 on page 232?

Q 8.34 Show that the shortcut Formula 8.10 works for portfolio N, in which H is 3/4. That is, does it give the same 79.3%% noted in Table 8.4?

Q 8.35 (This question is very important. Please do not pass over it.) Let's consider a stock market index, such as the S&P 500. It had a historical average rate of return of about 12% per annum, and a historical standard deviation of about 20% per annum. Assume for the moment:

(a) **Known statistical distributions:** You know the expected reward and risk. In our example, we assume that they are the historical averages and risks. This is convenient.

(b) **Independent stock returns:** Stock returns are (mostly) uncorrelated over time periods. This is reasonable because if this were not so, you could earn money purchasing stocks based on their prior performance in a perfect market. (This will be the subject of Chapter 11.)

(c) **No compounding:** The rate of return over X years is the simple sum of X annual rates of return. (That is, we ignore the cross-product terms that are rates of return on rates of return.) This is problematic over decades, but not over just a few months or even years.

Our goal is to work out how asset risk grows with time under these assumptions. The variance shortcut formula will help us.

(a) Write down the formula for the total rate of return over 2 years.
(b) What is the expected total rate of return over 2 years?
(c) Write down the formula for the variance over 2 years.
(d) What is the specific risk here (variance and standard deviation) over 2 years?
(e) The **Sharpe ratio** is a common (though flawed) measure of portfolio performance. It is usually computed as the expected rate of return above the risk-free rate, then divided by the standard deviation. Assume that the risk-free rate is 6%. Thus, the 1-year Sharpe ratio is $(12\% - 6\%)/20\% \approx 0.3$. What is the 2-year Sharpe ratio?
(f) What are the expected rate of return and risk (variance and standard deviation) over 4 years? What is the 4-year Sharpe ratio?
(g) What are the expected rate of return and risk (variance and standard deviation) over 16 years? What is the 16-year Sharpe ratio?
(h) What are the expected rate of return and risk (variance and standard deviation) over T years? What is the T-year Sharpe ratio?
(i) What are the expected rate of return and risk (variance and standard deviation) over 1 month? What is the 1-month Sharpe ratio?
(j) What are the expected rate of return and risk (variance and standard deviation) over 1 trading day? What is the 1-day Sharpe ratio? Assume 250 trading days per year.

8.8 GRAPHING THE MEAN-VARIANCE EFFICIENT FRONTIER

Graphing the trade-off between risk and reward . . .

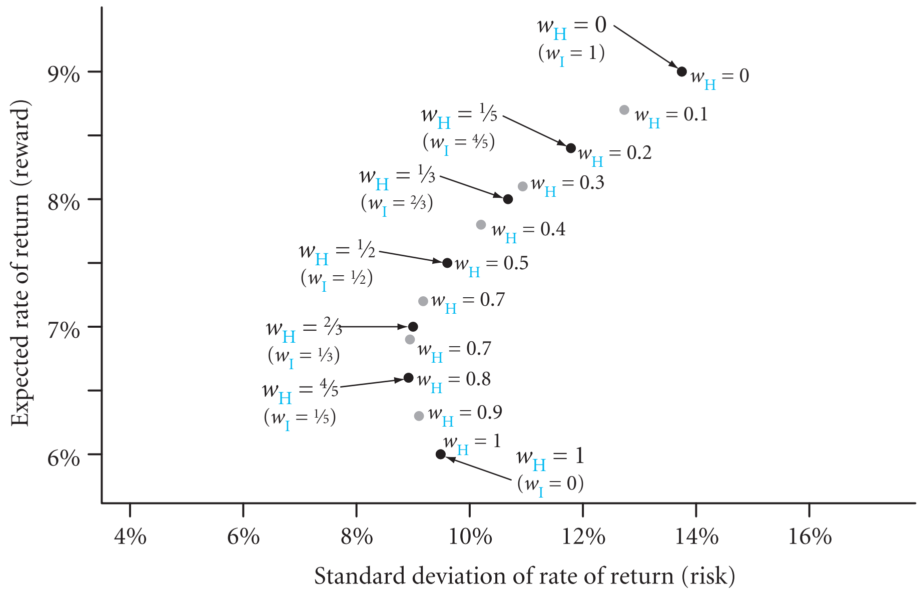
Let's now graph the portfolio risk on the x-axis and the portfolio reward on the y-axis for each portfolio from Table 8.4 on page 232. Figure 8.5 does it for you. Can you see a pattern? To make it easier, I have taken the liberty of adding a few more portfolios. (You can confirm that I have computed the risk and reward of one of these portfolios in Q 8.36.)

. . . is called the mean-variance efficient frontier.

If you picked many more portfolios with portfolio weights on H between 0 and 100%, you would eventually end up with Figure 8.6. The curve is called the **mean-variance efficient frontier** (**MVE frontier**), and it is the region where the best risk-reward portfolios lie. There must not be any portfolios to the northwest of this frontier—they would have a higher expected rate of return for a given risk, or lower risk for a given expected rate of return. If these existed, they would themselves be the MVE frontier. (The shape of the mean-variance efficient frontier is a so-called hyperbola when the x-axis is the standard deviation.)

The minimum-variance portfolio is the west-most portfolio.

The west-most portfolio on the efficient frontier is called the **minimum-variance portfolio** because you cannot create a portfolio with lower risk. You need a lot of algebra to find it, so I have worked this out for you. In our example, the minimum-variance portfolio has a weight of 76.191% on H and 23.809% on I, and it achieves



These are the portfolios from Table 8.4, and then some more in gray that I computed—a hobby.

FIGURE 8.5 The Risk-Reward Trade-Off between H and I: More Portfolios

as low a risk as 8.9%. Although the graph's scale is too small for you to check this graphically, you can compute the risk of this minimum-variance portfolio that I gave you and compare it to the risk of two portfolios that invest either a little more or a little less into H.

$$w_H = 76.0\% : \mathcal{S}dv(\tilde{r}_P) \approx 8.9042911\%$$

$$w_H = 76.2\% : \mathcal{S}dv(\tilde{r}_P) \approx 8.9042526\% \qquad \longleftarrow \text{I claimed lowest risk} \tag{8.11}$$

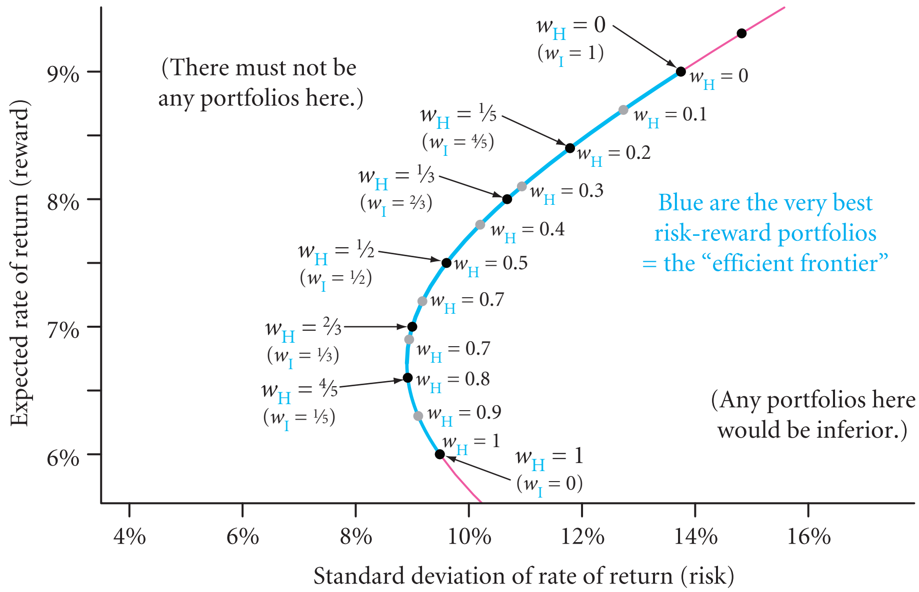
$$w_H = 76.4\% : \mathcal{S}dv(\tilde{r}_P) \approx 8.9042992\%$$

$$\mathcal{S}dv(\tilde{r}_P) = \sqrt{\mathcal{V}ar(\tilde{r}_P) = \mathcal{V}ar[w_H \cdot \tilde{r}_H + (1 - w_H) \cdot \tilde{r}_I]} \tag{8.12}$$

If there are assets that can be combined to construct a risk-free asset, then the minimum-variance portfolio will touch the y-axis at 0. If there are only two assets, this means their correlation would have to be −1. More commonly, the minimum-variance portfolio does not touch the y-axis and still has positive risk.

There is one feature of a more general mean-variance graph that this particular graph cannot illustrate. If you had started with more than two base portfolios H and I, you could have found many combination portfolios that would have been outright inferior. They would have been a cloud of points inside and southeast of the efficient frontier. However, the efficient frontier itself would still look very similar to what is in Figure 8.6—a hyperbola on the upper northwest frontier.

One feature is not visible in this figure, because there are only two portfolios: With more assets, many portfolio combinations lie in the interior—a cloud of points.



This connects the points on the efficient frontier to Figure 8.5. Additionally, it completes the efficient frontier beyond interior portfolios, that is, allowing for portfolios that short one or the other portfolio (in magenta).

FIGURE 8.6 The Risk-Reward Trade-Off between H and I: Sets

Allowing Shorted Positions

Extending the MVEF to allow for short positions extends the graph.

Each point on the mean-variance frontier represents one set of investment weights. Interestingly, the relevant formulas work just as well with negative weights as they do with positive weights. For example, if $w_H = (-0.1)$ and $w_I = 1.1$, then the sum of your individual investments is still 100%, and

$$\mathcal{E}(\tilde{r}_P) = (-0.1) \cdot 6\% + (1.1) \cdot 9\% = 9.3\%$$

$$\mathcal{E}(\tilde{r}_P) = w_H \cdot \mathcal{E}(\tilde{r}_H) + w_I \cdot \mathcal{E}(\tilde{r}_I)$$

and

$$\mathcal{S}dv(\tilde{r}_P) = \sqrt{(-0.1)^2 \cdot 90\%\% + (1.1)^2 \cdot 189\%\% + 2 \cdot (-0.1) \cdot (1.1) \cdot 45\%\%} \approx 14.82\%$$

$$\mathcal{S}dv(\tilde{r}_P) = \sqrt{w_H^2 \cdot \mathcal{V}ar(\tilde{r}_H) + w_I^2 \cdot \mathcal{V}ar(\tilde{r}_I) + 2 \cdot w_H \cdot w_I \cdot \mathcal{C}ov(\tilde{r}_I, \tilde{r}_H)}$$

(If you wish, you can first confirm this: This portfolio would return −12.6% (♣), 18.6% (♦), 26.4% (♥), or 4.8% (♠). Therefore, the expected rate of return is 9.3%, and the standard deviation is 14.82%.) This portfolio is marked at the top in Figure 8.6. It is on the continuation of the hyperbola. Actually, I have done more, drawing the rest of the hyperbola in magenta. These are portfolios that contain shorted assets.

The economic meaning of shorting reexplained.

➤ Shorting stocks, Section 7.2A, p. 191

But what is the meaning of an investment with negative weight? It was explained in Section 7.2A: It is shorting a stock. In brief, perfect shorting works as follows: If you short a security, you promise to provide the appropriate returns, rather than earn them. For example, say you want to go short \$200 in H and I want to go long \$200 in H. I would purchase H from you. This would work as follows:

- I must give you $200 today. (If you want, you can invest this to earn interest.)
- Next year, you must give me exactly what I would get if I had purchased H, not from you, but from someone else who really would have given me the security. That is, if ♣ comes about, you must pay me $188; if ♦ comes about, you must pay me $224; if ♥ comes about, you must pay me $200; and if ♠ comes about, you must pay me $236.

In other words, I won't notice whether you sold me the security or someone else (who had it) sold me the security. This is simple ownership—a 100% investment ownership. Your own rate of return is the exact opposite of my return. For example, if I earn −6%, you would gain +6%. After all, you received $200 from me (at time 0) and are only returning $188 to me (at time 1). What would your return be if you sold $200 of H to me, thereby going short, and then used the $200 to purchase H from someone else in the market? It would always be zero—going long and short by the same amount cancels out perfectly. In a perfect market, you would not earn any money or lose any money.

SOLVE NOW!

Q 8.36 Compute the risk and reward of the portfolio $w_H = 0.1$, $w_I = 0.9$, as in Table 8.4 on page 232. Confirm that this portfolio is drawn correctly in Figure 8.5.

Q 8.37 If there are two risky portfolios that have a correlation of −1 with positive investment weights, what would the expected rate of return on this portfolio be?

Q 8.38 If H and I were more correlated, what would the efficient frontier between them look like? If H and I were less (or more negatively) correlated, what would the efficient frontier between them look like? (Hint: Think about the variance of the combination portfolio that invests half in each.)

Q 8.39 Draw the efficient frontier for the following two base assets, H and Z:

	In Scenario			
Base Portfolio	S1 ♣	S2 ♦	S3 ♥	S4 ♠
H	−6%	+12%	0%	+18%
Z	−12%	+18%	+15%	+15%

Also, compute the covariance between H and Z. Is it higher or lower than what you computed in the text for H and I? How does the efficient frontier compare to what you have drawn in this chapter?

8.9 ADDING A RISK-FREE ASSET

In the real world, you usually have access to a risk-free Treasury. It turns out that the presence of a risk-free asset plays an important role, not only in the model of the next chapter (the CAPM), but also in these mean-variance graphs. So let us now add a risk-free rate ("F") of 4%. Start with the following three basis portfolios:

A special case: The risk and reward of combinations of portfolios with the risk-free asset are both simple linear functions.

Future	H	I	F
In Scenario S1 ♣	−6.0%	−12.0%	4.00%
In Scenario S2 ♦	+12.0%	+18.0%	4.00%
In Scenario S3 ♥	0.0%	+24.0%	4.00%
In Scenario S4 ♠	+18.0%	+6.0%	4.00%
"Reward" ($\mathcal{E}(R)$)	6.00%	9.00%	4.00%
"Variance" ($\mathcal{V}ar(R)$)	90.0%%	189.0%%	0.0%%
"Risk" ($\mathcal{S}dv(R)$)	9.49%	13.75%	0.00%

Begin by determining the risk and reward of a portfolio S that invests 1/2 in H and 1/2 in F: Its rate of return is defined as $\tilde{r}_S = w_H \cdot \tilde{r}_H + w_F \cdot \tilde{r}_F = 1/2 \cdot \tilde{r}_H + 1/2 \cdot 4\%$. The expected reward of this portfolio is

$$\begin{aligned} \mathcal{E}(\tilde{r}_S) &= 1/2 \cdot 6\% + (1 - 1/2) \cdot 4\% = 5\% \\ \mathcal{E}(\tilde{r}_S) &= w_H \cdot \mathcal{E}(\tilde{r}_H) + (1 - w_H) \cdot r_F \end{aligned} \tag{8.13}$$

➤ Portfolio variance, Formula 8.10, p. 234

For the risk component, use Formula 8.10. A risk-free rate, such as the 4% Treasury rate, has neither a variance nor a covariance with anything else. (Makes sense that a fixed constant number that is always the same has no variance, doesn't it?) For portfolio S, use $(1 - w_H) = w_F$ and you get

$$\begin{aligned} \mathcal{V}ar(\tilde{r}_S) &= (1/2)^2 \cdot 90\%\% + (1 - 1/2)^2 \cdot 0\% + 2 \cdot 1/2 \cdot (1 - 1/2) \cdot 0\% = \quad 1/4 \cdot 90\%\% \\ \mathcal{V}ar(\tilde{r}_S) &= \quad w_H^2 \cdot \mathcal{V}ar(\tilde{r}_H) + w_F^2 \cdot \mathcal{V}ar(r_F) + 2 \cdot w_H \cdot w_F \cdot \mathcal{C}ov(\tilde{r}_H, r_F) \quad = (w_H)^2 \cdot \mathcal{V}ar(\tilde{r}_H) \end{aligned}$$

This formula is a lot simpler than the typical variance formula, with its second variance term and its covariance term. It also means that we can compute the standard deviation more easily:

$$\begin{aligned} \mathcal{S}dv(\tilde{r}_S) &= \sqrt{(1/2)^2 \cdot 90\%\%} = 1/2 \cdot \sqrt{90\%\%} \approx 1/2 \cdot 9.49\% \approx 4.74\% \\ \mathcal{S}dv(\tilde{r}_S) &= \sqrt{(w_H)^2 \cdot \mathcal{V}ar(\tilde{r}_H)} = w_H \cdot \sqrt{\mathcal{V}ar(\tilde{r}_H)} = w_H \cdot \mathcal{S}dv(\tilde{r}_H) \end{aligned} \tag{8.14}$$

This states that the risk of your overall portfolio is proportional to the risk of your investment in asset H, with your investment weight being the proportionality factor.

Combining different weights of any risky portfolio with the risk-free asset yields a straight line.

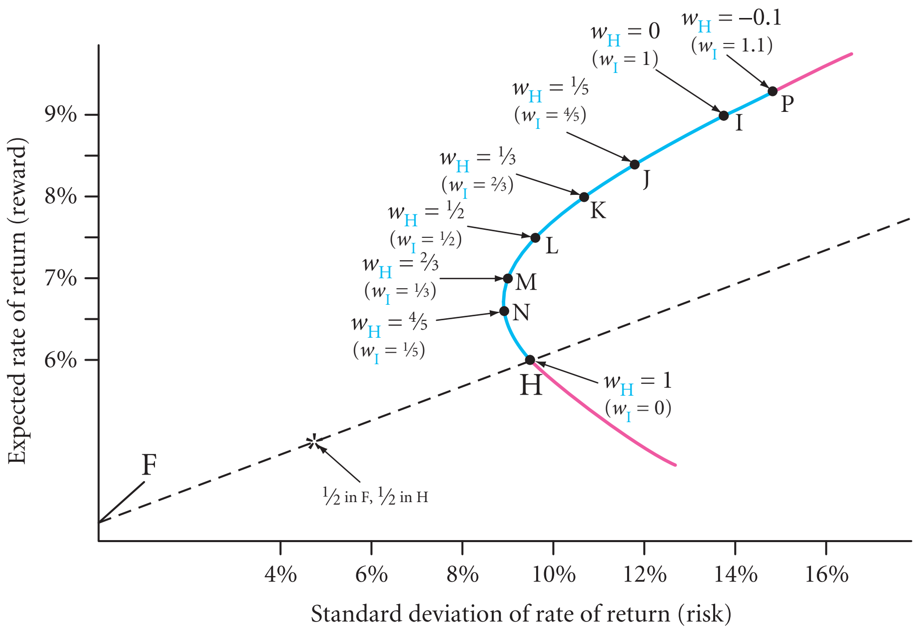
You can repeat this for many different portfolio weights:

Weight w_H	0.0	0.2	0.4	0.6	0.8	1.0
Expected Return	4.0%	4.4%	4.8%	5.2%	5.6%	6.0%
Standard Deviation	0.000%	1.898%	3.796%	5.694%	7.592%	9.490%

If you plot these points into the figure, you will immediately notice that the relationship between risk and reward is now a line. Figure 8.7 does it for you.

The algebra that shows that the relation between the risk and reward of a risky portfolio and the risk-free asset is a line.

You can also show this algebraically. Rearrange Formula 8.14 into $w_H = \mathcal{S}dv(\tilde{r}_S)/\mathcal{S}dv(\tilde{r}_H) = \mathcal{S}dv(\tilde{r}_S)/9.49\%$. Then use this to substitute out w_H in Formula 8.13:



This adds a risk-free rate of 4% to Figure 8.5. The line represents risks and rewards for portfolios that combine portfolio H and the risk-free rate F. Please note that this line is *not* the security markets line (the CAPM). Here, the *x*-axis is the standard deviation (of the overall portfolio rate of return). In the security market line (SML) explained in chapter 9, the *x*-axis is the market beta (of individual assets).

FIGURE 8.7 The Risk-Reward Trade-Off between H and F

$$\mathcal{E}(\tilde{r}_S) = w_H \cdot 6\% + (1 - w_H) \cdot 4\% = w_H \cdot (6\% - 4\%) + 4\%$$

$$= \left[\frac{\mathcal{S}dv(\tilde{r}_S)}{9.49\%}\right] \cdot (6\% - 4\%) + 4\% = 4\% + 0.21 \cdot \mathcal{S}dv(\tilde{r}_S)$$

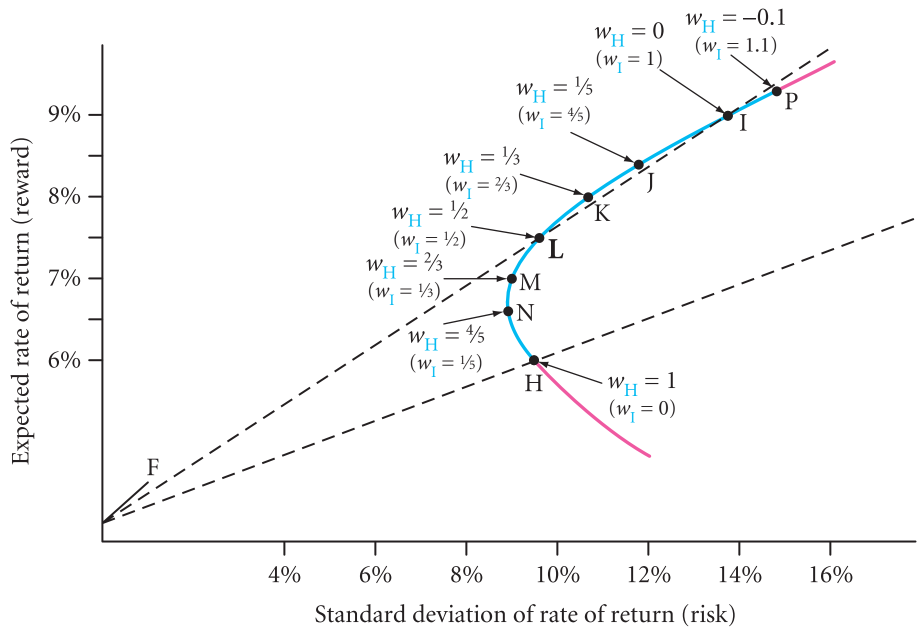
$$\mathcal{E}(\tilde{r}_S) = w_H \cdot \mathcal{E}(\tilde{r}_H) + (1 - w_H) \cdot r_F = w_H \cdot (\mathcal{E}(\tilde{r}_H) - r_F) + r_F$$

$$= \left[\frac{\mathcal{S}dv(\tilde{r}_S)}{\mathcal{S}dv(\tilde{r}_H)}\right] \cdot [\mathcal{E}(\tilde{r}_H) - r_F] + r_F = r_F + \left[\frac{\mathcal{E}(\tilde{r}_H) - r_F}{\mathcal{S}dv(\tilde{r}_H)}\right] \cdot \mathcal{S}dv(\tilde{r}_S)$$

This is the formula for a line: r_F is the intercept and $\left[(\mathcal{E}(\tilde{r}_H) - r_F)/(\mathcal{S}dv(\tilde{r}_H))\right]$ is the slope.

IMPORTANT: When you plot the portfolio mean versus the portfolio standard deviation for combination portfolios of a risk-free asset F with any risky portfolio P, they lie on the straight line between F and P.

But would you really want to purchase such a combination of H and F? Could you purchase a different portfolio in combination with F that would do better? Would the combination of L and F not perform better?



Adding to Figure 8.7, the new line represents risks and rewards for portfolios that combine portfolio L and the risk-free asset F.

FIGURE 8.8 The Risk-Reward Trade-Off between L and F

Figure 8.8 draws combinations of the risk-free asset and portfolio L. This combination of F and L indeed does a lot better—but you can do even better yet. Can you guess what portfolio you would purchase?

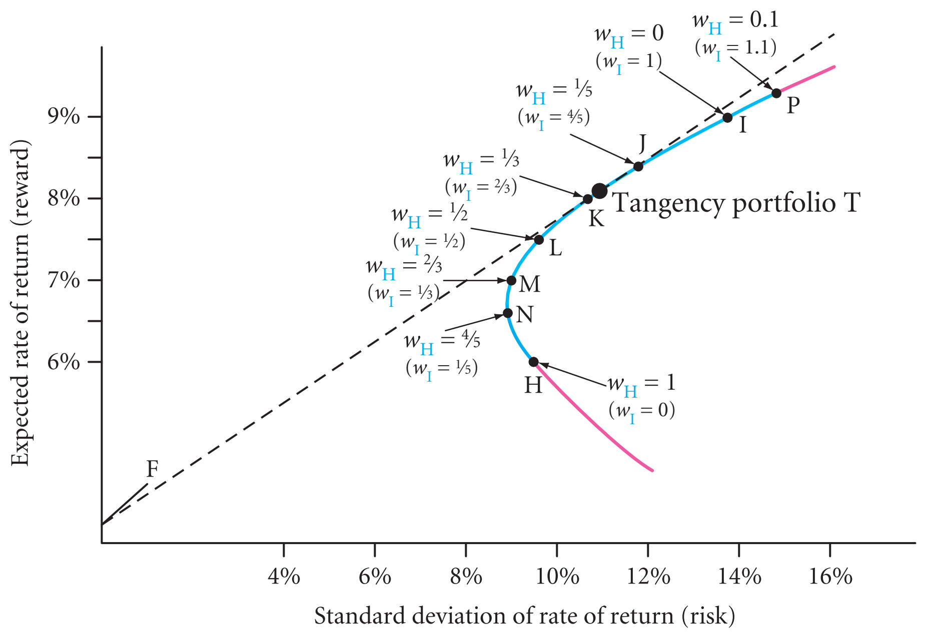
With a risk-free asset, the best portfolio is the line that is tangent to the efficient frontier of risky assets.

The answer is drawn in Figure 8.9—you would purchase a combination portfolio of the risk-free asset and whatever portfolio on the previous efficient frontier would be *tangent*—you tilt the line up until it just touches the mean-variance frontier among the risky assets. This line is called the **capital market line**. Here, the exact investment proportions in the risky assets are difficult to see, but if you could blow up the figure, you would see that this is the portfolio that invests about 30% in H and 70% in I. Let's call it T, for tangency portfolio.

Do all smart investors make the same portfolio decision in the presence of a risk-free asset? Yes and no.

Who would want to purchase a portfolio combination that invests more or less than 30% in H and 70% in I? Nobody! Each and every smart investor would purchase only a combination of F and T, regardless of risk aversion. (This is called the **two-fund separation theorem**.) Different risk tolerances would lead them to allocate different sums to the tangency portfolio and the risk-free asset, but no investor would purchase a risky portfolio with investment weights different from those in the tangency portfolio T.

IMPORTANT: In the presence of a risk-free asset, all smart investors purchase combinations of the tangency portfolio and the risk-free asset.



The capital market line represents risks and rewards for portfolios that combine the tangency portfolio T and the risk-free rate F. It represents the best opportunities available.

FIGURE 8.9 The Risk-Reward Trade-Off between T and F

CAPM Preview

Chapter 9 explains the most common model of security pricing, the CAPM. In brief, it states that the market portfolio is mean-variance efficient—and nothing else. How can this happen? Well, if *every* investor is smart and all the various CAPM assumptions and conditions are satisfied (explained soon), then each investor holds only a combination of T and the risk-free asset. Math dictates that this means that the value-weighted market portfolio of all investors' holdings is therefore also a combination of T and the risk-free asset. Therefore, it is also mean-variance efficient.

In equilibrium, if all investors buy combinations of the risk-free asset and the tangency portfolio, the market portfolio is on the efficient frontier.

IMPORTANT: In the CAPM, the market portfolio of risky claims is the tangency portfolio.

(Of course, conversely, if some investors do not hold the market tangency portfolio, then the overall market portfolio [could but] need not be the tangency portfolio.)

If the CAPM holds, that is, if T is the market portfolio, then portfolio optimization is beautifully easy for any investor—just purchase a combination of the market portfolio and the risk-free asset. You never even need to compute an efficient frontier. Of course, in the real world, the market portfolio may not be the tangency portfolio—but then, this is the same as stating that the CAPM does not hold. *In fact, the CAPM is nothing more and nothing less than the statement that the market portfolio is the tangency portfolio.*

The CAPM can make investing really easy—no computer program necessary!

SOLVE NOW!

Q 8.40 What kind of portfolios are the points to the right of H on the line itself in Figure 8.7?

Q 8.41 Compute the covariance of H and F.

Q 8.42 Formula 8.11 noted that the minimum-variance portfolio without a risk-free asset invests about 76.2% in H and about 24.8% in I. (Work with the rounded numbers to make your life easier.) With the risk-free asset offering 4%, what portfolio would you purchase that has the same risk, and what would its improvement in reward be? First think about how to solve this. However, this is a difficult question, so we will go through it step by step.

(a) Copy down the risk of this minimum-variance portfolio when there is no risk-free asset.
(b) What is the reward of this minimum-variance portfolio?
(c) With a risk-free rate of 4%, it turns out that the tangency portfolio invests 30% in H and 70% in I. What are its returns in each of the four scenarios?
(d) What is its reward? (Check this visually in the graph!)
(e) What is its risk? (Check this visually in the graph!)
(f) Using the analog of Formula 8.14, what investment weight w_T in T would give you the same risk as the minimum-variance portfolio? (If you had \$100, how much would you put into T, and how much would you put into a risk-free savings account?)
(g) Given this weight w_T, what is the reward of this combination portfolio? How much better is this than the situation where no risk-free asset was available?

Q 8.43 Would the tangency portfolio invest in more or less H if the risk-free rate were 3% instead of 4%? (Hint: Think visually.)

KEY TERMS

SOLVE NOW! SOLUTIONS

Q 8.31 The rates of return of portfolio M in Table 8.4 are −8% (♣), +14% (♦), 8% (♥), and 14% (♠). The deviations from the mean are −15%, 7%, 1%, and 7%. When squared, they are 225%%, 49%%, 1%%, and 49%%. The sum is 324%%; the average is 81%%. Thus, the standard deviation is indeed 9%.

Q 8.32 The portfolio variance of portfolio N in Table 8.4 is

$$\mathcal{S}dv(\tilde{r}_{\text{H}}) = \sqrt{\mathcal{V}ar(\tilde{r}_{\text{H}})} = \sqrt{\frac{(-7.5\% - 6.75\%)^2 + (13.5\% - 6.75\%)^2 + (6\% - 6.75\%)^2 + (15\% - 6.75\%)^2}{4}}$$

$$= \sqrt{\frac{203.0625\%\% + 45.5625\%\% + 0.5625\%\% + 68.0625\%\%}{4}}$$

$$\approx \sqrt{79.31\%\%} \approx 8.91\%$$

Q 8.33 For M, the covariance between H and I was computed as 45%% in Formula 8.9. The variance of H is 90%% (from Table 8.4 on page 232), the variance of I is 189%% (from the same figure). Therefore, using the shortcut Formula 8.10, $\mathcal{V}ar(\tilde{r}_{\text{M}}) = (2/3)^2 \cdot 90\%\% + (1/3)^2 \cdot 189\%\% + 2 \cdot (2/3) \cdot (1/3) \cdot 45\%\% = 81\%\%$.

Q 8.34 The covariance between H and I is 45%% (Formula 8.9). The variance of H is 90%%, the variance of I is 189%% (Table 8.4). Therefore, the shortcut Formula 8.10 gives

$$\mathcal{V}ar(\tilde{r}_{\text{M}}) = (3/4)^2 \cdot 90\%\% + (1/4)^2 \cdot 189\%\% + 2 \cdot (3/4) \cdot (1/4) \cdot 45\%\% = 79.3125\%\%$$

Q 8.35 This is an important question. In fact, you should memorize Formula 8.15 that describes how risk grows over time. The assumption that there is no compounding (that you can ignore the cross-product) and that risk is roughly constant per period is reasonable over periods that are not more than a few years long.

(a) If we can ignore the cross-products, then we are using a simple weighted-average formula with weights of 1 on each term: $\tilde{r}_{0,2} \approx 1 \cdot \tilde{r}_{0,1} + 1 \cdot \tilde{r}_{1,2}$. (The exact formula would have been $\tilde{r}_{0,2} = \tilde{r}_{0,1} + \tilde{r}_{1,2} + \tilde{r}_{0,1} \cdot \tilde{r}_{1,2}$.)

(b) The expected rate of return over 2 years is $\mathcal{E}(\tilde{r}_{0,2}) \approx \mathcal{E}(\tilde{r}_{0,1}) + \mathcal{E}(\tilde{r}_{1,2}) = 12\% + 12\% = 24\%$.

(c) The variance of the rate of return over 2 years is $\mathcal{V}ar(\tilde{r}_{0,2}) \approx 1 \cdot \mathcal{V}ar(\tilde{r}_{0,1}) + 1 \cdot \mathcal{V}ar(\tilde{r}_{1,2}) + 2 \cdot 1 \cdot 1 \cdot \mathcal{C}ov(\tilde{r}_{0,1}, \tilde{r}_{0,2})$. In a perfect market, the last term should be approximately zero.

(d) The variance over 2 years for our specific example is

$$\mathcal{V}ar(\tilde{r}_{0,2}) \approx 1 \cdot \mathcal{V}ar(\tilde{r}_{0,1}) + 1 \cdot \mathcal{V}ar(\tilde{r}_{1,2}) + 0$$

$$= (20\%)^2 + (20\%)^2 = 2 \cdot (20\%)^2 = 800\%\%$$

Therefore, the standard deviation is $\sqrt{2} \cdot 20\% \approx 28\%$.

(e) The Sharpe ratio is $2 \cdot (12\% - 6\%)/28\% \approx 0.43$.

(f) The variance is $4 \cdot (20\%)^2 = 1600\%\%$. The standard deviation is $20\% \cdot \sqrt{4} = 40\%$. The Sharpe ratio is $(6\% \cdot 4)/(20\% \cdot \sqrt{4}) = 0.3 \cdot \sqrt{4} = 0.6$.

(g) The variance is $16 \cdot (20\%)^2 = 6400\%$. The standard deviation is $20\% \cdot \sqrt{16} = 80\%$. The Sharpe ratio is $0.3 \cdot \sqrt{16} \approx 1.2$.

(h) The variance is $T \cdot (20\%)^2$. The standard deviation is $20\% \cdot \sqrt{T}$. In other words, the standard deviation grows with the square root of the number of time periods:

$$\textbf{IMPORTANT: How asset risk grows with time:} \quad \mathcal{S}dv(\tilde{r}_{0,T}) \approx \sqrt{T} \cdot \mathcal{S}dv(\tilde{r}_{0,1}) \qquad (8.15)$$

If the rates of return on an asset are approximately uncorrelated over time (a perfect market consequence), if the risk in different time periods remains constant, and ignoring all cross-product terms. The Sharpe ratio is $0.3 \cdot \sqrt{T}$.

(i) The formulas also work with fractions. The variance is therefore $1/12 \cdot (20\%)^2 \approx 33.3\%\%$. The standard deviation is therefore $\sqrt{1/12} \cdot 20\% \approx 5.8\%$. The monthly Sharpe ratio is $\sqrt{1/12} \cdot 30\% \approx 0.09$.

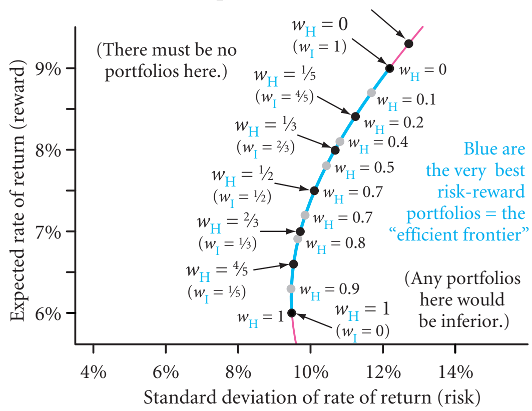
(j) The variance is $1/250 \cdot (20\%)^2 = 1.6\%\%$. The standard deviation is $\sqrt{1/250} \cdot 20\% \approx 1.3\%$. The daily Sharpe ratio is about 0.019.

Q 8.36 The mean rate of return for portfolio ($w_H = 0.1$, $w_I = 0.9$) is $0.1 \cdot 6\% + 0.9 \cdot 9\% = 8.7\%$. You can also compute this from the rates of return in the 4 states −11.4%, 17.4%, 21.6%, and 7.2%. De-meaned, these returns are −20.1%, 8.7%, 12.9%, and −1.5%. The variance is therefore $(404.01\%\% + 75.69\%\% + 166.41\%\% + 2.25\%\%)/4 = 162.09\%\% = 0.016209$. Therefore, the standard deviation (risk) is $\sqrt{162.09\%\%} \approx 12.7\%$.

Q 8.37 Two risky portfolios with a correlation of −1 can be combined into an asset that has no risk. Thus, its expected rate of return has to be the same as that on the risk-free asset—or you could get rich in a perfect market.

Q 8.38 If the correlation was higher, diversification would help less, so the risk would be higher. Therefore, the efficient frontier would not bend as far toward the west (a risk of 0). An easy way to check this is to rearrange the returns so that they correlate more positively, as you will do in the next question. If the correlation was lower, diversification would help more, so the risk would be lower. Therefore, the efficient frontier would bend closer toward the west (a risk of 0).

Q 8.39 The covariance between H and Z is 85.5%%, which is much higher than the 45%% covariance between H and I from Formula 8.9 on page 233. This means that the correlation between H and Z shoots up to 74% (from 35% for the correlation between H and I). This means that the efficient frontier is less dented toward the west. Put differently, the minimum-variance portfolio moves toward the east.



Q 8.40 Portfolios to the right of H on the line have a negative weight in F and a weight above 1 in H. (The portfolio weights must add to 100%!) This means that they would borrow money at a 4% annual interest rate to purchase more of portfolio H. (Purchasing stocks with money borrowed at an interest rate is called **on margin**.)

Q 8.41 Because the net-of-mean F is always 0, so is its coproduct with anything else. This means that the covariance of the risk-free asset with any risky asset is zero, too.

Q 8.42 This question asks you to show how much better off you are with this particular risk-free asset for a particular risk choice.

(a) In Formula 8.12 on page 237, we showed that this no-risk-free minimum-variance portfolio with an investment weight of 76.2% in H and 24.8% in I has a risk of about 8.90%.

(b) The reward of this no-risk-free-asset-available, minimum-variance portfolio is $\mathcal{E}(\tilde{r}) = 76.2\% \cdot 6\% + 24.8\% \cdot 9\% \approx 6.8\%$.

(c) With a weight of 30% in H and 70% in I, the rates of return in the four scenarios for the tangency portfolio T are as follows:

$$\text{In Scenario } \clubsuit: \quad 0.3 \cdot (-6\%) + 0.7 \cdot (-12\%) = -10.2\%$$
$$\text{In Scenario } \diamondsuit: \quad 0.3 \cdot (12\%) + 0.7 \cdot (18\%) = +16.2\%$$
$$\text{In Scenario } \heartsuit: \quad 0.3 \cdot (0\%) + 0.7 \cdot (24\%) = +16.8\%$$
$$\text{In Scenario } \spadesuit: \quad 0.3 \cdot (18\%) + 0.7 \cdot (6\%) = +9.6\%$$

(These calculations will reappear later in Table 9.2 on page 290.)

(d) The reward of the tangency portfolio is $\mathcal{E}(\tilde{r}_T) = (-10.2\% + 16.2\% + 16.8\% + 9.6\%)/4 = 8.1\%$.

(e) Its risk is $\mathcal{S}dv(\tilde{r}_T) = \sqrt{[(-18.3\%)^2 + (8.1\%)^2 + (8.7\%)^2 + (1.5\%)^2]/4} \approx 10.94\%$.

(f) You want the expected rate of return of a portfolio that uses the risk-free asset and that has a risk of 10.94% (i.e., the same that the no-risk minimum-variance portfolio had). Solve

$$8.9\% = w_T \cdot 10.94\%$$

$$\mathcal{S}dv(\tilde{r}) = w_T \cdot \mathcal{S}dv(\tilde{r}_T)$$

Therefore, $w_T \approx 81.35\%$. In words, a portfolio of 81.35% in the tangency portfolio T and 18.65% in the risk-free asset F has the same risk of 10.94%.

(g) You now want to know the expected rate of return on the portfolio $(w_T, w_F) = (81.35\%, 18.65\%)$:

$$\mathcal{E}(\tilde{r}) \approx 81.35\% \cdot 8.1\% + 18.65\% \cdot 4\% \approx 7.33\%$$

$$\mathcal{E}(\tilde{r}) = w_T \cdot \mathcal{E}(\tilde{r}_T) + w_F \cdot r_F$$

You therefore would expect to receive a $7.33\% - 6.71\% \approx 62$ basis points higher expected rate of return if you have access to this risk-free rate.

Q 8.43 If the risk-free rate were lower, then the tangency line would become steeper. The tangency portfolio would shift from around K to around L. Therefore, it would involve more H.

PROBLEMS

The [ ] *indicates problems available in* myfinancelab

Q 8.44 Recompute the portfolio variance if you invest in a portfolio O with $w_H = 90\%$ and $w_I = 10\%$ in Table 8.4.

(a) Compute the rates of return on the portfolio in each scenario, and then treat the resulting portfolio as one asset. What is portfolio O's risk and reward?

(b) Compute the same variance with the shortcut Formula 8.10 on page 234.

Q 8.45 An asset has an annual mean of 12% and standard deviation of 30% per year. What would you expect its monthly mean and standard deviation to be?

Q 8.46 Mathematically and based on Figure 8.6 on page 238, the risk and reward of the portfolio $w_H = -0.2$, $w_I = -1.2$.

Q 8.47 In the absence of a risk-free asset, would anyone buy the portfolio $w_H = 110\%$, $w_I = -10\%$?

Q 8.48 The Vanguard European stock fund, Pacific stock fund, and Exxon Mobil reported the following historical dividend-adjusted prices:

Year	1991	1992	1993	1994	1995	1996
VEURX	6.53	7.15	6.91	9.34	9.03	11.17
VPACX	7.18	7.41	6.30	9.52	9.08	9.97
XOM	9.57	10.07	10.88	10.97	15.29	19.18

Year	1997	1998	1999	2000	2001
VEURX	13.50	17.45	21.42	23.38	23.13
VPACX	8.39	7.17	7.01	10.41	8.10
XOM	24.63	30.14	33.94	37.42	34.57

Year	2002	2003	2004	2005	2006
VEURX	17.50	14.42	21.22	24.87	29.53
VPACX	5.64	5.42	7.94	9.08	11.93
XOM	31.50	38.01	48.67	54.41	75.67

(a) Compute the means and covariances of the rates of return on these three assets.

(b) Draw the efficient frontier if you can only invest in VEURX and VPACX.

(c) Now add Exxon Mobil. Use Excel to draw 1,000 random numbers in two columns, called wE and wP. (Create one formula, and copy it into all of the cells.) Each of these 2,000 cells should use the formula 'rand()*3-1'. Create a new column that is 1.0 minus wE and wP, and call it wX. Now consider these random numbers as investment weights in VEURX, VPACX, and XOM. Compute the risk and reward for each of these portfolios (one portfolio is three numbers: one wE, one wP, and one wX), using the standard deviation and expected rate of return formulas. Finally, create an *x*-*y* plot that shows, for each of your wE, wP, and wX portfolios, the risk-reward combinations. What does the plot look like?

(d) If the risk-free rate stood at 5% per annum, what would be the tangency portfolio?

Q 8.49 Return to the example with a risk-free asset in Formula 8.14 on page 240. What are the risk and reward of a portfolio that invests $w_H = 150\%$? (This means that if you have $100, you would borrow $50 at the 4% annual interest rate to purchase $150 of H—more than your portfolio wealth itself.)